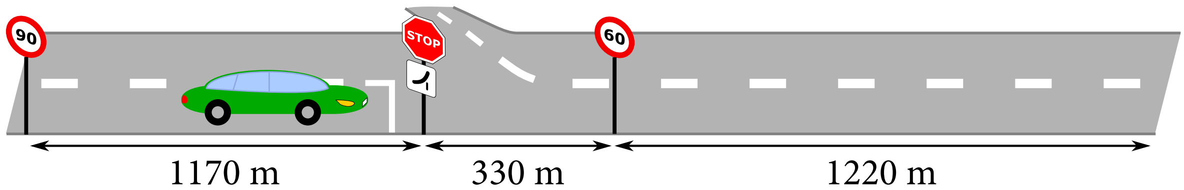
Problemy

1 Marcin kupił nowy samochód, który przyspiesza od 0 do 100 km/h w 10 sekund i zwalnia od 100 km/h do 0 w 6 sekund. W jakim najkrótszym czasie Marcin przejedzie drogę pokazaną na obrazku, jeśli będzie przestrzegać wszystkich przepisów ruchu drogowego?

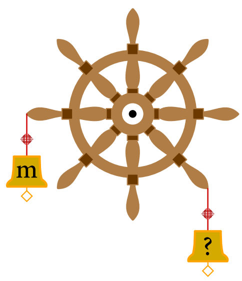
Oczywiście na początku i na końcu drogi jego samochód nie porusza się. Ograniczenia prędkości na znakach drogowych podano w km/h.



2 Beata znalazła bardzo fajny odcinek autostrady biegnący wzdłuż linii kolejowej. Udaje się tam czasami, aby ścigać się z pendolino. Pociąg ten ma 150 razy większy pęd, ale tylko 100 razy większą energię kinetyczną niż samochód Beaty. Jaki jest stosunek masy pociągu pendolino do masy samochodu?

3 Spacerując po targu, Marcin spostrzegł duży balon wypełniony helem. Jego promień wynosił 50 cm, a masa pustego balonu była równa 20 g. Balon bardzo mu się spodobał i natychmiast go kupił… ale zaczął się obawiać, że mógłby mu on odlecieć. Pomyślał, że przywiązując do niego coś ciężkiego, mógłby zapobiec jego bezpowrotnej stracie. Zdecydował się nadmuchać kilka mniejszych balonów powietrzem i przywiązać je do balonu z helem. Masa każdego małego, pustego balonu była równa 10 g, a jego promień po nadmuchaniu był równy 20 cm. Ile, co najmniej, balonów z powietrzem musiałby użyć Marcin, aby uniemożliwić odlecenie balonu z helem? Pomiń rozciągliwość balonów.

4 Monika uczestniczyła w rejsie, z którego jako pamiątkę przywiozła ozdobny ster. W celu przymocowania go do ściany, wbiła gwóźdź i umieściła na nim środek steru. Ponieważ nie miała wystarczającej ilości miejsca na ścianie na nowe pamiątki, zawiesiła chiński dzwonek o masie m po lewej stronie steru. Ster się nieco obrócił, więc musiała zawiesić kolejny dzwonek po jego prawej stronie. Jaka powinna być masa drugiego dzwonka, aby koło powróciło do pierwotnej pozycji?



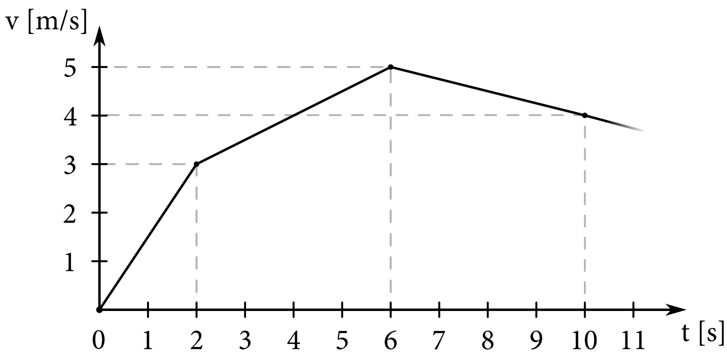
5 Dwóch Szkotów walczy o monetę o wartości jednego grosza. Każdy z nich ciągnie jeden koniec monety, do momentu uformowania się z niej drutu o promieniu r, długości l i oporze R. Nie powstrzymuje ich to jednak od dalszej walki. Jaki będzie opór drutu, gdy podwoi on swoją długość?

Załóż, że objętość monety nie zmienia się z powodu rozciągania.

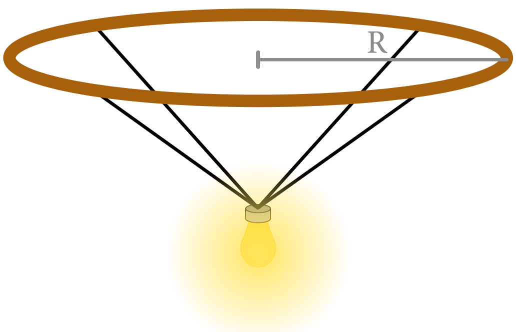
6 Sheldon lubi flagi. Postanowił uszyć własną flagę Seszeli. Kiedy skończył, zauważył, że jej środek masy nie znajduje się dokładnie w geometrycznym środku flagi. Ważąc kawałki materiału, których użył do każdej części flagi, ustalił, że gęstości tkanin od lewej do prawej strony wynosiły odpowiednio 600, 400, 300, 500 i 900 g/m^2. W jakiej odległości od geometrycznego środka flagi znajduje się jej środek masy, jeśli posiada ona rozmiary 180 cm × 90 cm?

Szwy dzielą górną i prawą krawędź flagi na trzy równe części.

7 Michał ciągnie Alicję na sankach, używając stałej siły F. Sanie wraz z Alicją mają masę $m = 60$ kg. Na początku Michał ciągnie sanie po zamarzniętym jeziorze, gdzie nie występuje tarcie. Następnie ciągnie je po śniegu, gdzie współczynnik tarcia między śniegiem a saniami wynosi $f > 0$. Po chwili na sanie wskakuje Piotrek. Prędkość sań pokazana jest na wykresie. Jaka jest masa Piotra?



8 Marcin przeniósł się do nowego, pustego mieszkania. Będąc astronomem nie miał dużo pieniędzy. Postanowił, że część potrzebnych do życia rzeczy wykona sam. Postanowił więc skonstruować własny żyrandol. Jego żyrandol zbudowany był z obręczy o promieniu$R = 24$ cm, która zwisała z sufitu. Następnie przymocował do obręczy żarówkę o masie $m = 1$ kg używając czterech jednorodnych, nieważkich nici. Mając bardzo ograniczony budżet wykorzystał niezbyt mocne nici. Wytrzymałość na zerwanie była równa $F = 6{,}5$ N. W jakiej najmniejszej odległości od obręczy może powiesić żarówkę, aby nici nie uległy zerwaniu?



9 Andrzej postawił w Bratysławie wysoki maszt, na którego szczycie powiewa monumentalna flaga o powierzchni 10 m^2. Jest południe równonocy jesiennej. Wiatr wieje z północnego zachodu. Jaka jest powierzchnia cienia rzucanego na ziemię przez całkowicie rozwiniętą flagę?

Bratysława znajduje się na 48 równoleżniku szerokości geograficznej północnej.

10 Karol podróżował pociągiem, aby odwiedzić swoich dziadków. Kiedy się nudził, liczył samochody, które mijały pociąg na autostradzie biegnącej wzdłuż linii kolejowej. Zaobserwował on cztery razy więcej samochodów na jednostkę czasu jadących w przeciwnym kierunku niż w kierunku swojej podróży. Jakie były możliwe prędkości, z którymi poruszał się pociąg Karola?

Ruch na autostradzie był równomierny w obu kierunkach, a prędkość samochodów wynosiła 90 km/h.

11 Dwa księżyce, każdy o masie m, poruszają się po tej samej orbicie kołowej wokół planety o masie $2m$. W każdej chwili wszystkie trzy obiekty leżą na jednej prostej. Wykonując pomiary Kasia stwierdziła, że promień orbity księżyców wynosił R. Ile wynosi okres obiegu księżyców wokół planety?

12 W ciepłe letnie noce bezpańskie koty bardzo hałasują i tym samym nie dają spać mieszkańcom akademików. Pewnej, szczególnie gorącej nocy, Tomek nie mogąc zasnąć, stracił panowanie nad sobą, wyszedł na balkon, który znajdował się na wysokości H nad ziemią i zaczął rzucać w koty zgniłymi jajkami. Wszystkie jajka miały tę samą prędkość początkową v, ale były rzucane pod różnymi kątami α.

Jaka jest najmniejsza prędkość, jaką może uzyskać jajko w momencie uderzenia o ziemię? Opory powietrza należy pominąć.

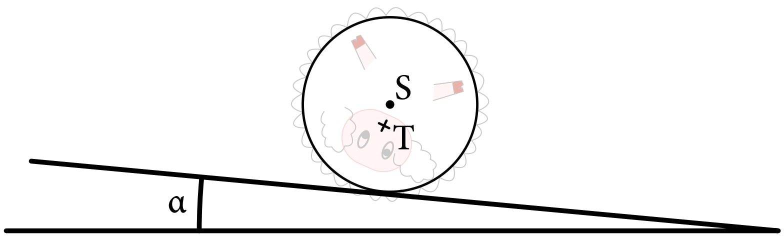
13 Piotr i Paweł uzyskali niedawno patenty żeglarskie. Wynajęli dwa jachty i rozpoczęli podróż przez siedem mórz. Jednak ich umiejętności żeglarskie nie były zbyt duże, dlatego ze względów bezpieczeństwa płynęli tylko do przodu ze stałą prędkością.

- Gdy Piotr przeciął drogę Pawła ich wzajemna odlegóść wynosiła $3d$.
- Po czasie t Paweł przeciął drogę Piotra, wtedy ich wzajemna odległość wynosiła $4d$.
- Po kolejnym odcinku czasu równym $5t$ skończyło się im paliwo i natychmiast się zatrzymali, a ich wzajemna odległość wynosiła $21d$.

Pod jakim kątem przecinały się ich drogi?

14 Odkąd Jerzy otrzymał tytuł magistra fizyki, jego zwyczaje pasterskie uległy zmianie i wyglądają dość dziwnie w ocenie pozostałych pasterzy. Jerzy z upodobaniem (będąc fizykiem teoretykiem), modeluje swoje owce jako jednorodne walce o promieniu R i masie m. Kiedy owca kończy wypas na wzgórzu, ma w zwyczaju podciągać nogi i staczać się z niego.

Nie jest to jednak takie łatwe jakby się wydawało. Gdy owca zje dostatecznie dużo trawy, jej środek masy T przesuwa się o $R/4$ od środka S wzdłuż promienia. Jaki jest najmniejszy kąt nachylenia wzgórza, aby owca z pewnością stoczyła się z niego, niezależnie od początkowego kąta ułożenia ciała (walca)?



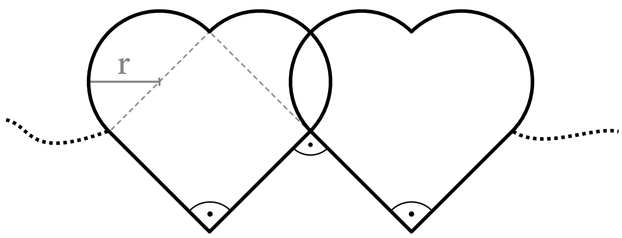
15 Mała księżniczka Leia siedzi na swojej małej planecie P-314 i jest bardzo zdenerwowana. Gwiazda jej planety zachodzi co 60 godzin, co oznacza, że aby zobaczyć wystarczająco dużo romantycznych zachodów słońca, musi ciągle przesuwać swoje ciężkie drewniane krzesło. Z drugiej jednak strony rok ma tylko 300 godzin, co pozwala jej świętować wiele urodzin.

Po jednym z przyjęć urodzinowych przesunęła krzesło na nocną stronę planety i spojrzała w gwiazdy. Po chwili zdała sobie sprawę, że odległe gwiazdy wydają się krążyć wokół planety z innym okresem obiegu niż jej słońce. Jaki jest faktyczny okres obrotu planety wokół własnej osi?

Załóżmy, że planeta krąży wokół gwiazdy w tym samym kierunku, w którym się obraca, i że jej orbita jest okręgiem. Oś obrotu jest prostopadła do płaszczyzny orbity.

16 Sławek kupił Ani na urodziny naszyjnik złożony z dwóch takich samych, symetrycznych serc. Ania była oczarowana, ale ponieważ jest fizykiem, użekła ją nie tylko estetyka, ale także elektryczne właściwości biżuterii.

Dowiedziała się, że opór na jednostkę długości drutu, z którego wykonany jest naszyjnik, wynosi ρ. Jaki jest całkowity opór serc między punktami, w których są one połączone z łańcuszkiem? Promień krzywizny okrągłych części naszyjnika wynosi r.



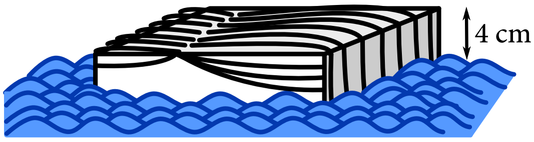
17 Łukasz lubi budować proste urządzenia gospodarstwa domowego. Ostatnio zakupił trzy rezystory o oporach 20, 30 i 60 Ω i zmontował z nich grzejnik elektryczny. Niestety nie jest on dobrym projektantem, więc kiedy chce zmienić moc grzejnika, najpierw musi go rozebrać, a następnie ponownie połączyć rezystory. Ile różnych ustawień mocy (nie włączając w to stanu wyłączenia) może uzyskać, jeśli główny wyłącznik prądu ustawiony jest na 15 A?

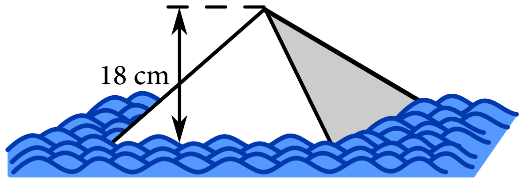
Napięcie sieciowe wynosi 230 V. Łukasz nie musi za każdym razem używać wszystkich rezystorów.

18 Krystian ma piękny zegar ścienny. Ma on tylko wskazówkę godzinową i wskazówkę minutową, które przemieszczają się skokowo co sekundę. Wskazówka godzinowa ma długość d i masę $2m$, natomiast wskazówka minutowa ma długość $2d$ i masę m. O której godzinie w ciągu doby suma momentów sił wskazówek, liczonych względem osi przechodzącej przez środek tarczy zegara, będzie maksymalna?

Znajdź przynajmniej jedno rozwiązanie. Wynik podaj w postaci `HH:MM:SS`. *Nie obawiaj się używać kalkulatora.*

19 Sześcian Darek bardzo lubi pływać w basenie. Jeśli leży on tak, że jego podstawa jest równoległa do powierzchni wody, wtedy wystaje 4 cm ponad powierzchnię lustra. Natomiast, jeśli się odwróci tak, aby jedna z przekątnych ciała była prostopadła do powierzchni wody, jego widoczną częścią jest trójstronna piramida wystająca 18 cm ponad powierzchnię lustra. Jaka jest gęstość Darka?





20 Mateusz podniósł kamień o masie m i położył go na przednim krańcu deskorolki o długości L i masie M. Pomiędzy deskorolką a podłożem nie występuje tarcie, a współczynnik tarcia między kamieniem a deskorolką wynosi μ. Mateusz pchnął deskorolkę tak, że zaczęła się ona poruszać z prędkością v. Jaka jest najwyższa możliwa prędkość v deskorolki, aby kamień na niej pozostał?

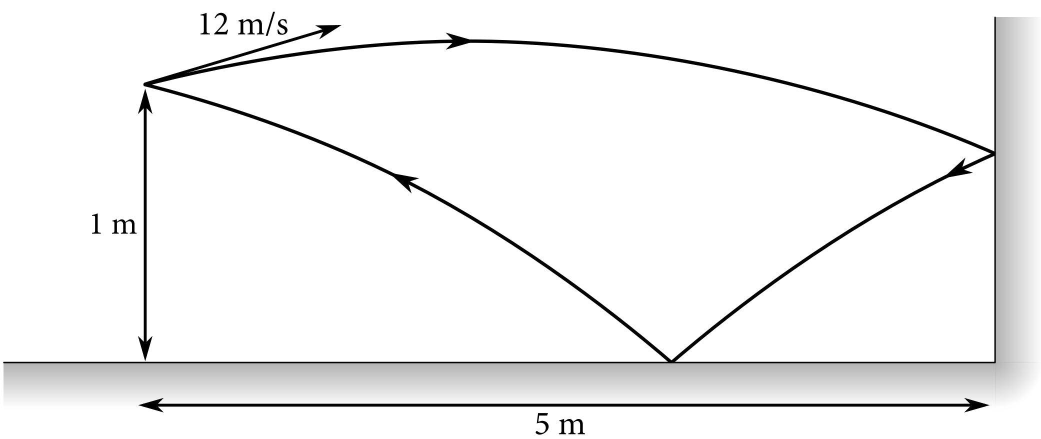
Załóżmy, że deskorolka została przyspieszona natychmiast, a kamień w pierwszej chwili pozostawał w spoczynku względem Mateusza.

21 Tomek przypadkowo zrzucił ze stołu swój nowiutki smartfon, o wysokości ekranu $l = 16$ cm. Na szczęście w ostatniej chwili udało mu się wprowadzić go w ruch obrotowy. Ponieważ posiadał nowoczesny telefon z wyświetlaczem obejmującym całą przednią ścianę, jego jedyną nadzieją było to, że telefon wyląduje bezpośrednio na tylnej ścianie. W przeciwnym razie wyświetlacz prawdopodobnie uległby kosztownemu uszkodzeniu. Wysokość stołu wynosiła $h = 0{,}8$ m, a telefon pierwotnie leżał na tylnej ścianie. Ile maksymalnie obrotów mógłby wykonać telefon w powietrzu, aby wylądował bezpiecznie na tylnej ścianie, tak jak chciałby tego Tomek?

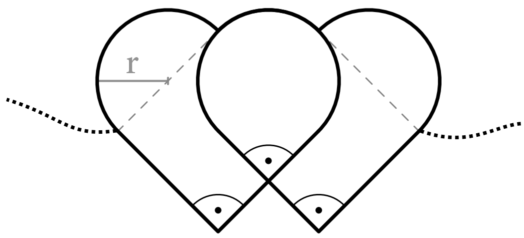
Załóżmy, że smartfon nie dostał żadnego dodatkowego impulsu w kierunku pionowym podczas rozpoczynania obrotów.

22 Paulina wzięła dwa lustra i postawiła je na stole, tak aby kąt między ich powierzchniami odbijającymi wynosił α. Następnie skierowała wskaźnik laserowy między lustra tak, że wiązka była równoległa do jednego z nich. Zauważyła, że wiązka powróciła do niej wzdłuż dokładnie tej samej trajektorii, w jakiej została puszczona. Jakie są możliwe wartości kąta α między lustrami, aby taka sytuacja mogła zaistnieć?

23 W kieszeni swoich spodni Marek niespodziewanie znalazł sprężystą piłeczkę. Bez zastanowienia rzucił ją z prędkością $v = 12$ m/s z wysokości $h = 1$ m w stronę ściany, która znajdowała się w odległości $d = 5$ m. Piłeczka uderzyła w ścianę, a następnie odbiła się od podłogi i z powrotem trafiła do ręki Marka. Jak długo trwała cała podróż piłeczki, jeśli wszystkie jej odbicia były idealnie sprężyste?



24 Ania była tak zachwycona prezentem otrzymanym od Sławka, że teraz kocha go jeszcze bardziej. W związku z tym postanowiła wprowadzić kilka zmian w naszyjniku. Nadal składa się on z dwóch takich samych, symetrycznych serc, opór na jednostkę długości materiału wynosi ρ, a promień krzywizny okrągłych fragmentów r. Jaki jest całkowity opór serc między punktami, w których są one połączone z łańcuszkiem?

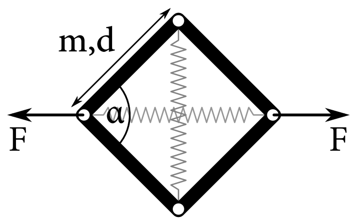


25 Mateusz, będąc organizatorem zawodów fizycznych „Nabój", przyniósł dla uczestników w dużym metalowym pojemniku w kształcie cylindra, napełnionego do połowy, ciepłą herbatą. Pojemnik miał zamontowany w dnie kran, dzięki któremu istniała możliwość przelania herbaty do mniejszych naczyń. Wysokość cylindra wynosi 1 m. Mateusz postanowił przelać herbatę do kubków, jednak po chwili zauważył, że przestała się ona wylewać z pojemnika, ponieważ do środka nie mogło dostać się powietrze. Jak bardzo powierzchnia herbaty może opaść, jeżeli nie wpuścimy powietrza do środka pojemnika? Powietrze wewnątrz cylindra przed otwarciem kranu, miało standardowe ciśnienie atmosferyczne.

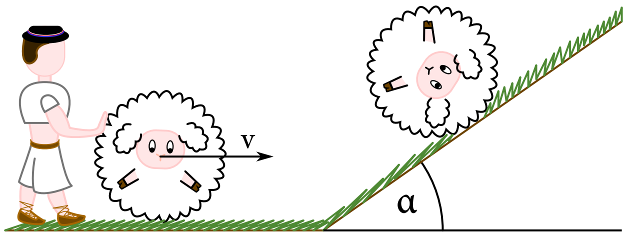
Wynik podaj z dokładnością do 1 mm.

26 Michalina uwielbia tworzyć systemy mechaniczne za pomocą sprężyn. Kupiła kwadratową ramkę, której każdy bok ma masę m i długość d. Boki ramki są połączone za pomocą specjalnych kostek w rogach, co sprawia, że podczas wyginania cała ramka pozostaje w tej samej płaszczyźnie.

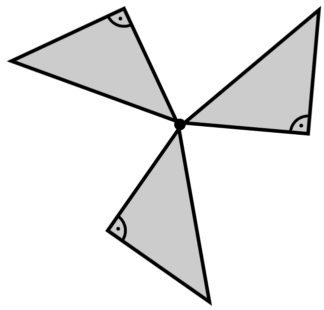
Michalina położyła ramkę poziomo na ziemi. Następnie wzięła dwie sprężyny o współczynniku sprężystości k i rozciągnęła je wzdłuż przekątnych ramki. Potem zaczęła odsuwać od siebie dwa przeciwległe rogi. Jaką siłą F musi ciągnąć rogi, aby kąt przy jej dłoni wyniósł α?



27 Kiedy Jerzy zorientował się, że owce bezpiecznie staczają się ze wzgórza, wpadł na kolejny pomysł, jak możnaby wykorzystać ich cylindryczny kształt. Toczył owce po płaskiej powierzchni z prędkością v, aż do podnóża wzniesienia o kącie nachylenia α. Do jakiej maksymalnej wysokości trawa na wzgórzu położy się, aż do zatrzymania się owiec? Promień głodnej owcy wynosi r.

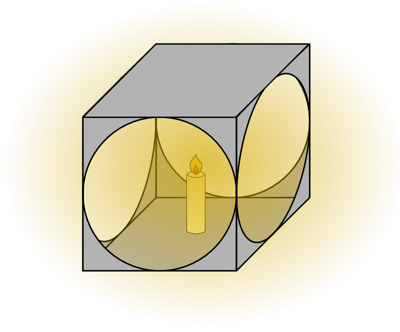


28 Kasia bada alternatywne źródła energii. W ostatnim czasie kupiła w bardzo okazyjnej cenie wiele arkuszy blachy o gęstości powierzchniowej σ i skonstruowała z nich proste śmigło. Składa się ono z trzech łopatek, każde w kształcie równoramiennego trójkąta o długości ramienia a. Łopatki są przymocowane do piasty równomiernie co 120°. Jaki jest moment bezwładności zbudowanego przez Kasię śmigła względem osi piasty?



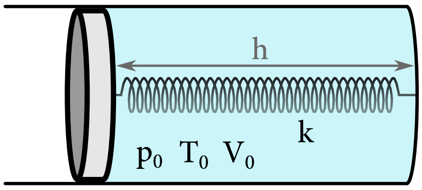
29 Horacy znalazł sprężynę o długości 1 m. Stosując pewne sekretne metody, wyznaczył współczynnik sprężystości tej sprężyny, który był równy 70 N/m. Postanowił zbudować platformę. W tym celu pociął sprężynę na kilka mniejszych, równych kawałków tak, aby móc położyć na nich drewnianą deskę o masie 10 kg. Na ile kawałków musi Horacy pociąć sprężynę, by stając na desce był utrzymywany nad podłożem? Masa Horacego wynosiła 100 kg.

30 Tomek miał duży bałagan w pokoju. Nieoczekiwania znalazł w nim magiczną latarnię. Ma ona sześcienny kształt, a światło jest emitowane przez cztery otwory w bocznych ścianach, które mają kształt okręgów wpisanych w kwadrat. Jaki kąt bryłowy może oświetlić latarnia, jeśli źródło światła znajduje się dokładnie w jej środku?



31 Oliwier wziął długi cylindryczny pojemnik i zamknął go tłokiem. Przymocował tłok do dna pojemnika za pomocą sprężyny, której współczynnik sprężystości wynosi k (sprężyna nie jest rozciągnięta). W pojemniku znajdował się gaz doskonały o objętości V_0, ciśnieniu p_0 i temperaturze T_0.

W pewnym momencie zdał sobie sprawę, że stopień wydłużenia (rozciągnięcia) sprężyny h, może być wskaźnikiem stanu gazu doskonałego. Natychmiast narysował schemat h–T procesu adiabatycznego dla gazu znajdującego się w pojemniku. Założył, że żadne siły zewnętrzne nie mogą oddziaływać na tłok, a jedynymi siłami działającymi na niego są: siła parcia gazu wewnątrz pojemnika i siła sprężystości sprężyny. Narysuj, podobnie jak Oliwier, odpowiedni schemat i zaznacz na nim wszystkie istotne parametry.



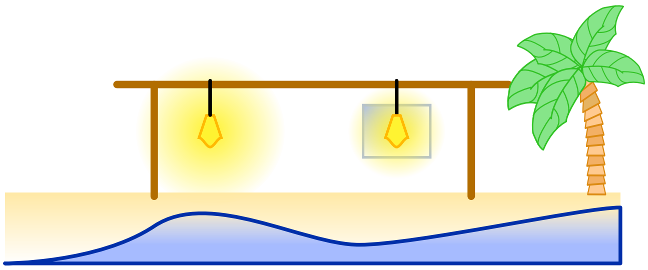
32 Szymon pilnie potrzebuje światła spolaryzowanego poziomo. Do dyspozycji ma źródło światła spolaryzowanego pionowo o natężeniu I_0 i 10 idealnych filtrów polaryzacyjnych. Chce ustawić filtry w taki sposób, aby obróciły płaszczyznę polaryzacji. Jakie jest największe natężenie światła spolaryzowanego poziomo za filtrami, jeśli światło nie może zawierać żadnego elementu spolaryzowanego pionowo?

Idealny filtr polaryzacyjny przepuszcza całe światło w kierunku polaryzacji i blokuje całe światło w kierunku prostopadłym do niego.

33 Adam postanowił zniszczyć pewną instytucję w swoim sąsiedztwie. Pomyślał, że najlepszym sposobem będzie wykorzystanie działa o długość lufy L i przekroju poprzecznym S. Następnie wsunął do lufy pocisk o masie m. W wyniku tego pozostałe za pociskiem powietrze miało objętość V_0 i ciśnienie p_0. Wybuch prochu dostarczył do powietrza energię E. W wyniku tego powietrze zaczęło się rozprężać adiabatycznie wyrzucając pocisk. Oblicz prędkość pocisku w momencie, gdy opuszczał on lufę. Załóż, że powietrze to doskonały gaz dwuatomowy.

34 Syreny Hanna i Nina próbują zwabić żeglarzy za pomocą światła swoich latarni (latarnie są identyczne). Nina zawiesiła swoją latarnię na uchwycie, skąd świeci ona izotropowo z mocą P. Hanna natomiast umieściła swoją latarnię w wielkim sześcianie wykonanym ze szkła o współczynniku załamania światła n i ustawiła ją tak, że jedna ze ścian była skierowana w stronę morza.

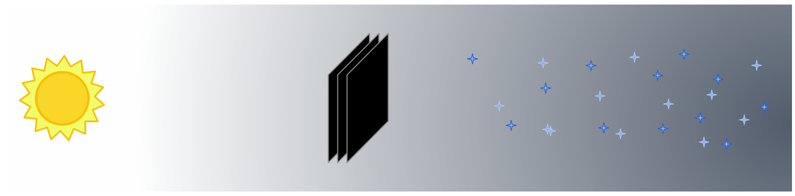
Która latarnia jest jaśniejsza dla marynarzy i ile razy, jeśli patrzą oni prostopadle do brzegu z dużej odległości?



35 Justyna bardzo lubi fizykę cząstek elementarnych. Zbudowała własną komorę mgłową (komora Wilsona) i umieściła ją w silnym polu magnetycznym o indukcji $B = 10$ T. Zauważyła, że w pewnej chwili w komorze zostały zarejestrowane ślady protonu i mionu. Miały one kształt dwóch współśrodkowych półokręgów. Obydwie cząstki poruszały się z tą samą prędkością $v = 0{,}99c$ prostopadłą do linii pola magnetycznego. Wektory prędkości cząstek przed wejściem do komory były równoległe. Jaka była wzajemna odległość między cząstkami wpadającymi w pole magnetyczne?

Mion ma ładunek równy ładunkowi elektronu i jest 207 razy cięższy od niego.

36 Za oknem stacji kosmicznej rozwieszono na linach jeden z drugim trzy duże, idealnie czarne koce. Słońce świeci prostopadle na pierwszy koc z mocą na jednostkę powierzchni $F_\odot = 1370\ \mathrm{W/m^2}$. Jaka jest temperatura równowagi trzeciego koca, jeśli za nim jest tylko pusta przestrzeń kosmiczna?



37 Mateusz postanowił się wybrać na spacer po znanym Podkarpackim Szlaku Winnym. Już w pierwszej piwnicy bardziej zainteresował się samymi beczkami niż ich zawartością. Miały one kształt cylindra o promieniu podstawy 0,5 m i wysokości 1,6 m. Zrobione były z prostych listewek spiętych metalowymi obręczami. Zaciekawiło go ile, co najmniej metalowych obręczy jest potrzebnych, aby stojąca i wypełniona całkowicie winem beczka nie rozpadła się.

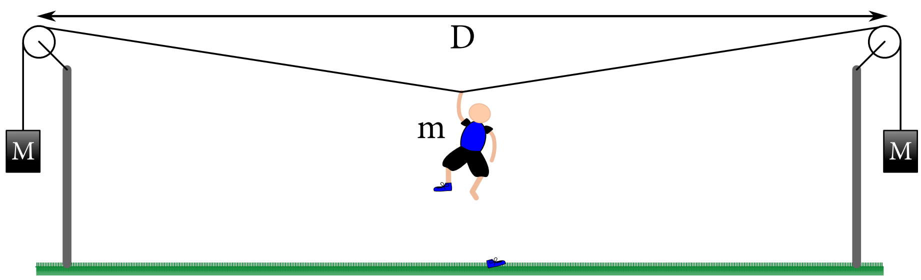
Maksymalne dopuszczalne naprężenie metalowej obręczy wynosi 20 MPa*, jej przekrój poprzeczny* 30 mm^2*, a gęstość wina równa jest* 1000 kg/m^3.

38 Szymon wciąż bawi się filtrami polaryzacyjnymi. Po wielu eksperymentach filtry trochę się ubrudziły. Teraz każdy z nich przepuszcza tylko 90% wpadającego w płaszczyźnie polaryzacji światła (ale nadal nie ma światła w płaszczyźnie prostopadłej).

Jakie jest największe natężenie światła spolaryzowanego poziomo, które Szymon może uzyskać po przejściu przez filtry? Padające światło nadal jest spolaryzowane pionowo, a jego natężenie wynosi I_0. Szymon ma 10 filtrów polaryzacyjnych, ale nie musi używać ich wszystkich.

39 Maria puszczała bańki mydlane. Jest ona osobą sympatyczną i o dobrych chęciach, zatem nie podobało się jej, że powietrze wewnątrz bańki było ściskane. Z tego powodu postanowiła rozmieścić na bańce ładunek elektryczny. Jaki ładunek powinna umieścić na bańce o promieniu 4 cm, by ciśnienie wewnątrz bańki było równe atmosferycznemu? Załóż, że napięcie powierzchniowe wody mydlanej jest równe 1/3 napięcia powierzchniowego czystej wody. Wynik podaj zaokrąglając do trzech cyfr znaczących.

40 Emil stale ulepsza swoją makietę kolejową. Ostatnio zajął się liniami elektrycznymi. Linę nośną zawiesza się przez parę kół pasowych umieszczonych u góry słupów i naciąga się ją ciężkimi obciążnikami przymocowanymi do jej końców. Podczas rutynowej kontroli swoich linii, Emil doznał tajemniczego wypadku, w wyniku którego huśtał się na środku liny zawieszonej między dwoma słupami. Znajdź okres drgań pionowych jego ruchu. Załóż, że $\frac{M}{m} = \frac{41}{18}$ i że słupy są oddalone od siebie o $D = 25$ m



Rozwiązania

1 First of all we express the acceleration and deceleration of the car in units that are much easier to use:

$$a_+ = \frac{100\ \text{km/h}}{10\ \text{s}} = 10\frac{\text{km/h}}{\text{s}} \qquad \text{a} \qquad a_- = \frac{100\ \text{km/h}}{6\ \text{s}} = \frac{100}{6}\frac{\text{km/h}}{\text{s}}.$$

To avoid manipulating ugly numbers, we keep accelerations in this form and we transform time so that it fits these units. Hence, when given acceleration and time, the expression at^2 takes units

$$\frac{\text{km/h}}{\text{s}} \cdot \text{h} \cdot \text{s} \equiv \frac{\text{km/h}}{\text{s}} \cdot 3600\ \text{s} \cdot \text{s}.$$

For clarity, let us divide the path to segments between traffic signs:

I. First we need to accelerate from rest to 90 km/h, which takes 9 s. The distance covered within this period is

$$s_{\text{Ia}} = \frac{1}{2}a_+t^2 = 112{,}5\ \text{m}.$$

We have to stop at the end of the first segment. Deceleration takes 5,4 s and the distance is

$$s_{\text{Ic}} = v_{\text{I}}t - \frac{1}{2}a_-t^2 = 60\ \text{km/h} \cdot \frac{5{,}4\ \text{s}}{3600\ \text{s/h}} - \frac{1}{2} \cdot \left(\frac{100}{6}\frac{\text{km/h}}{\text{s}}\right) \cdot \frac{(5{,}4\ \text{s})^2}{3600\ \text{s/h}} = 67{,}5\ \text{m}.$$

The remaining part of the first segment $s_{\text{Ib}} = 990$ m is travelled at constant speed. Therefore the resulting time to travel the first segment is

$$t_{\text{I}} = 9\ \text{s} + \frac{s_{\text{Ib}}}{v_{\text{I}}} + 5{,}4\ \text{s} = 54\ \text{s}.$$

II. The beginning is same as in the first segment. Namely, we accelerate in 9 s and we cover a distance of $s_{\text{IIa}} = 112.5$ m. At the end, we have to slow down to 60 km/h. It takes time 1,8 s and the car travels a distance of

$$s_{\text{IIc}} = 90\ \text{km/h} \cdot 1{,}8\ \text{s} - \frac{1}{2} \cdot \frac{100}{6}\frac{\text{km/h}}{\text{s}} \cdot (1{,}8\ \text{s})^2 = 37{,}5\ \text{m}.$$

The rest of the segment is travelled at constant speed, thus resulting time is

$$t_{\text{II}} = 9\ \text{s} + \frac{s_{\text{IIb}}}{v_{\text{I}}} + 1{,}8\ \text{s} = 18\ \text{s}.$$

III. Deceleration at the end of the ride takes 3,6 s and corresponding distance is

$$s_{\text{IIIb}} = v_{\text{III}}t - \frac{1}{2}a_-t^2 = 30\ \text{m}.$$

The remaining part of the segment is $s_{\text{IIIa}} = 1190$ m long. We pass it with constant speed, thus in the third segment we spend time

$$t_{\text{III}} = \frac{s_{\text{IIIa}}}{v_{\text{III}}} + 3{,}6\ \text{s} = 75\ \text{s}$$

Summing it all together, the total time of the drive is 147 s.

2 Denote the mass of the train M, the speed of the train V and mass and speed of Kate's car m and v, respectively. From the task, we know that the ratio of their momenta is

$$\frac{MV}{mv} = 150, \tag{2.1}$$

from which we can figure out the ratio of the speeds

$$\frac{V}{v} = 150\frac{m}{M}. \tag{2.2}$$

The ratio of the kinetic energies is

$$\frac{\frac{1}{2}MV^2}{\frac{1}{2}mv^2} = 100. \tag{2.3}$$

IF we substitute the ratio of speeds from the equation 2.2 into the equation 2.3, we obtain

$$\frac{M}{m} = 100\frac{v^2}{V^2} = \frac{100}{150^2}\frac{M^2}{m^2}, \qquad \text{from where} \qquad \frac{M}{m} = 225.$$

3 Let us denote the mass of the helium balloon M, smaller balloons m and their radii R and r respectively. The balloon cluster does not float away if its total weight is not smaller than the lifting force acting on it. Having realised that a weight of the air inside the smaller balloons balances the lifting force acting on smaller balloons, we get a condition for not floating away

$$M + Nm + \frac{4}{3}\pi R^3 \rho_{\mathrm{He}} \geq \frac{4}{3}\pi R^3 \rho_a.$$

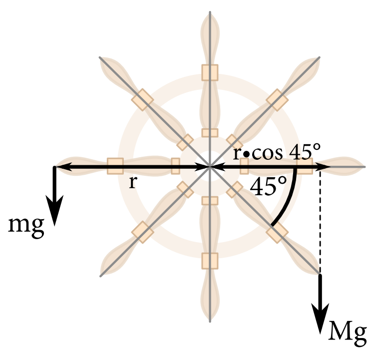
For the least number of balloons, we obtain

$$N = \left\lceil \frac{\frac{4}{3}\pi R^3 \left(\rho_a - \rho_{\mathrm{He}}\right) - M}{m} \right\rceil.$$

When we put in the numerical values, we obtain $N = 6$.

4 To prevent the ship's wheel from rotating, the total torque calculated with respect to the wheel's axis has to be zero. Therefore

$$mgr = Mgr\cos 45° \quad \Rightarrow \quad M = m\frac{1}{\frac{\sqrt{2}}{2}} = \sqrt{2}m.$$



5 A resistance of the wire depends on its length, cross-sectional area and material's resistivity ρ as

$$R = \rho \frac{l}{S}.$$

Plastic deformation of the wire change neither its volume nor specific resistivity. If the length of the wire is doubled due to pulling, the cross-sectional area decreases to one half because of conservation of volume. The resistance of the elongated wire is thus

$$R' = \rho \frac{2l}{\frac{S}{2}} = 4\rho \frac{l}{S} = 4R.$$

6 Denote area densities of individual triangles gradually from left σ_1 to σ_6. Their masses are calculated simply by multiplying the densities by their areas. We obtain

$$m_1 = 162\ \text{g}, \quad m_2 = 108\ \text{g}, \quad m_3 = 81\ \text{g}, \quad m_4 = 81\ \text{g}, \quad m_5 = 135\ \text{g}, \quad m_6 = 243\ \text{g}.$$

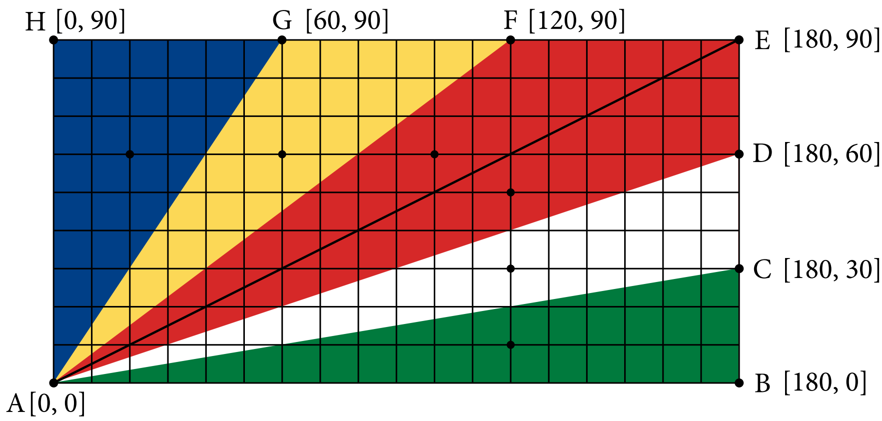


Figura 6.1: *Vlajka na štvorcovej sieti.*

Next, we find centres of mass of individual triangles. Let's begin with triangle $\triangle AGH$. The centroid, and thus also a centre of mass, of a triangle is at the intersection of its medians, which is in one third of any median measured from its base. If we place the triangle onto a regular grid, draw a median connecting vertex A with the midpoint of edge GH and find the point in two thirds of its length, we see that coordinates of the centroid are $[20\ \text{cm}; 60\ \text{cm}]$. Analogously we can find the coordinates of centroids of other triangles:

$$T_1 = [20\ \text{cm}; 60\ \text{cm}], \quad T_2 = [60\ \text{cm}; 60\ \text{cm}], \quad T_3 = [100\ \text{cm}; 60\ \text{cm}],$$

$$T_4 = [120\ \text{cm}; 50\ \text{cm}], \quad T_5 = [120\ \text{cm}; 30\ \text{cm}], \quad T_6 = [120\ \text{cm}; 10\ \text{cm}].$$

Consequently, we need to find the centre of mass of the whole flag. In general, the formula for the x-coordinate of the centre of mass of objects with masses $m_1, m_2, \ldots, m_n$ and with x-coordinates of their centroids $x_1, x_2, \ldots, x_n$ is

$$X = \frac{m_1 x_1 + m_2 x_2 + m_3 x_3 + \ldots + m_n x_n}{m_1 + m_2 + m_3 + \ldots + m_n}.$$

In our case, we obtain $X = 90$ cm.

Analogously, the y-coordinate of the centroid can be calculated:

$$Y = \frac{m_1 y_1 + m_2 y_2 + m_3 y_3 + \ldots + m_n y_n}{m_1 + m_2 + m_3 + \ldots + m_n} \doteq 39\ \mathrm{cm}.$$

The coordinates of the geometric centre of the flag are $x_S = 90$ cm and $y_S = 45$ cm, therefore the distance of the centroid of the flag from its geometric centre is approximately 6 cm.

7 Accelerations of the sled in the individual segments can be determined from the graph:

$$a_1 = 1{,}5\ \mathrm{m/s^2}, \qquad a_2 = 0{,}5\ \mathrm{m/s^2}, \qquad a_3 = -0{,}25\ \mathrm{m/s^2}.$$

In the first segment, Isaac pulls with force F. According to the second Newton's law

$$m = \frac{F}{a_1}. \tag{7.1}$$

In the second segment, a friction force $F_{t1} = fmg$ acts as well, where f is a friction coefficient between the sled and ground. The sum of forces acting on the sled is thus

$$F - F_{t1} = F - fmg = ma_2. \tag{7.2}$$

In the last segment, there are two differences – higher total mass $m + M$ and a higher friction force $F_{t2} = f(M+m)\,g$. We obtain

$$F - F_{t2} = F - f(M+m)\,g = (M+m)\,a_3. \tag{7.3}$$

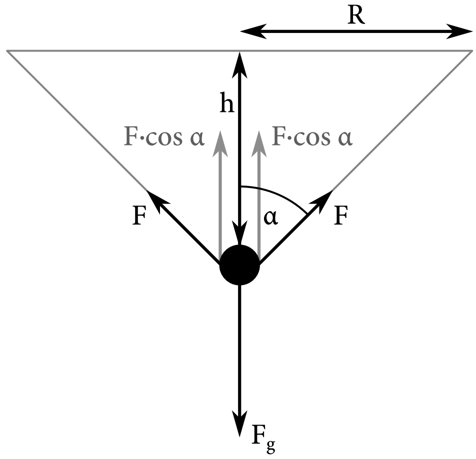
Now, we are able to determine everything we are interested in. From equation 7.1, we substitute for m in 7.2 and we obtain

$$f = \frac{a_1 - a_2}{g}.$$

Then we substitute for f in 7.3 and we finally obtain Peter's mass as

$$M = m\frac{a_3 - a_2}{a_2 - a_1 - a_3} = 60\ \mathrm{kg}.$$

8 Since the chandelier is radially symmetric, we can consider just a simpler two-dimensional problem:



It is clear that the vertical component of the tension force in each string is $F\cos\alpha$. In order to prevent the chandelier from falling down, the resulting force acting on the bulb must be zero. As there are four strings attached to the bulb, we obtain

$$4F\cos\alpha = mg \quad\Rightarrow\quad \alpha = \arccos\frac{mg}{4F}.$$

We want to know height h. It can be expressed from the equation above, as

$$\tan\alpha = \frac{R}{h} \quad\Rightarrow\quad h = \frac{mgR}{\sqrt{16F^2 - m^2g^2}}.$$

After plugging in the numeric values, we obtain $h = 10$ cm.

9 Firstly, we calculate the height of the Sun on the sky in Bratislava. At noon, the Sun shines directly from south. On the day of equinox, the Sun is directly above the equator. The closest point of the Earth to the Sun is on the intersection of the equator and the meridian passing through Bratislava. Light beams arrive in Bratislava with angle of $90° - 48° = 42°$ with respect to the local surface.

As the wind blows from north-west, the flag is oriented in south-eastern direction, thus it is rotated by 45° with respect to sunbeams. Therefore, the cross-sectional area of the flag is effectively smaller when looking from the Sun. If we rotated the flag to the east-west direction, we would need a flag with a crossectional area only $\cos 45° \cdot 10\ \mathrm{m}^2$ to shade the same amount of sunbeams, as is shown on the left half of picture 9.1.

The sunbeams are projected on the ground. The length of the shadow in the east-west direction remains unchanged, whilst it is elongated in the north-south direction by factor h'/h. We get

$$h = h'\tan 42° \quad\Rightarrow\quad h' = h\cot 42°.$$

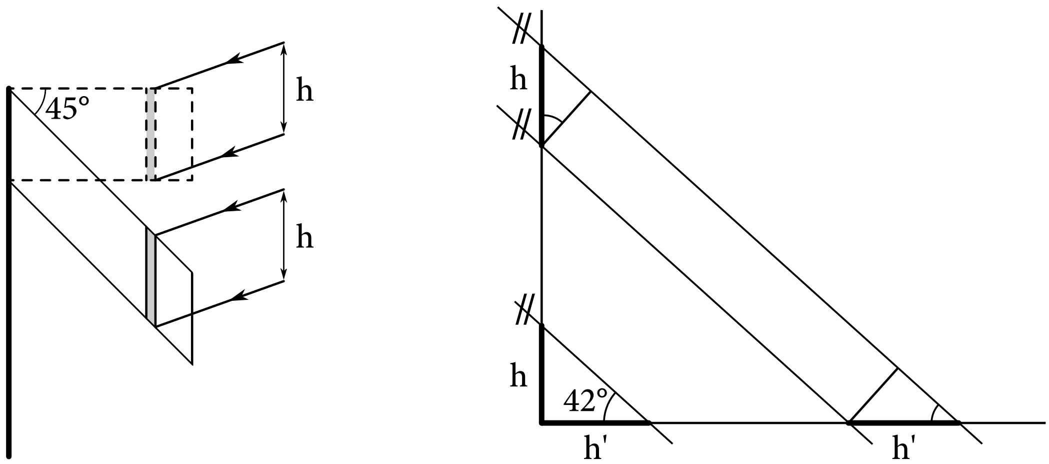


Figura 9.1: *The projection of the flag to east-west direction and to the ground.*

Therefore, the shadow on the ground has an area of $\frac{\cos 42°}{\sin 42°} \cdot \cos 45° \cdot 10\ \mathrm{m}^2 = \cot 42° \cdot \cos 45° \cdot 10\ \mathrm{m}^2 \doteq 7{,}85\ \mathrm{m}^2$.

Note that this is also valid for any non-rectangular shape of the flag – any shape can be arbitrarily well approximated by a large number of non-overlapping rectangles.

10 Denote an average distance between two consucutive cars d. Then an average time between meeting two consecutive cars is $t = \frac{d}{w}$, where w is a mutual velocity of cars and a train. A frequency is only a inverse value of the average time.

Let v be a velocity of cars and u a velocity of the train. Then a frequency of meeting cars driving in the opposite direction is $f^+ = \frac{v+u}{d}$ and cars driving in the same direction is $f^- = \frac{|v-u|}{d}$.[1] Obviously, $f^+ > f^-$. Suppose that f^+ is k-times greater than f^-. Then

$$v + u = k\,|v - u|\,.$$

We need to find a velocity of the train u satisfying the equation above. We have to consider two cases. If $v > u$, then $v + u = k\,(v - u)$, therefore

$$u = \frac{k-1}{k+1}v.$$

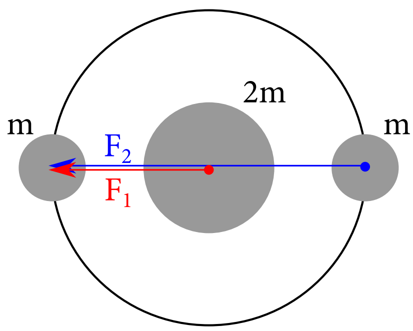
If $v < u$, then $v + u = k\,(u - v)$, thus

$$u = \frac{k+1}{k-1}v.$$

For $k = 4$ and $v = 90$ km/h, we obtain two possible velocities of the train $u = 54$ km/h and $u = 150$ km/h.

11 Each of the moons is attracted by gravitational force of the other moon and of the planet. In order to orbit the circular trajectory, the resulting force acting on the moon must be a cenripetal force corresponding to its radius R, e. i.

$$F_1 + F_2 = G\frac{Mm}{R^2} + G\frac{m^2}{4R^2} \stackrel{!}{=} \frac{mv^2}{R}.$$



In order to caculate the period, we need to express the speed of the moons. Consequently, the period is obtained as a ratio of covered distance and moon's speed

$$T = \frac{2\pi R}{v}.$$

Therefore

$$T = \frac{2\pi R}{\sqrt{\frac{9Gm}{4R}}} = \frac{4\pi R^{\frac{3}{2}}}{3\sqrt{Gm}}.$$

[1] An absolute value is present to secure that the frequency is always positive, as the train can be either faster or slower than cars.

12 During the throw, only the gravity acts on the egg, which is conservative – hence it can be described by a potential. Therefore, the problem can be solved in terms of kinetic and potential energy.

Thomas always throws the egg with same initial speed, regardless the direction. Therefore, a kinetic energy of the egg at tho moment of throwing is always

$$E_k(H) = \frac{1}{2}mv^2$$

and a potential energy is

$$E_p(H) = mgH$$

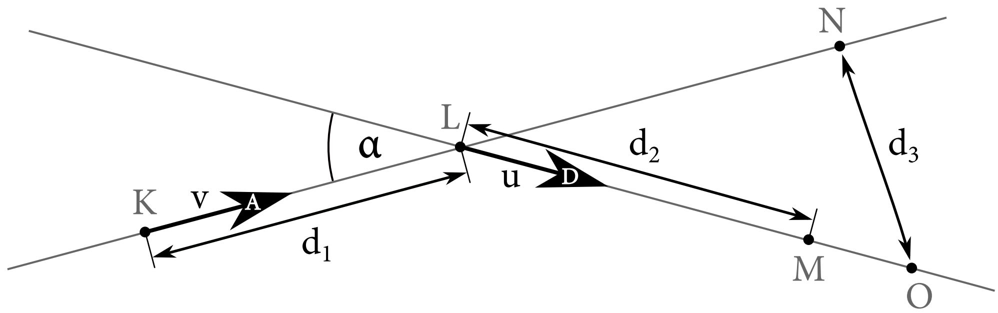
(considering a ground to be a level of zero potential energy).

A sum of the kinetic and potential energy is conserved, thus at the zero hight above the ground we get

$$E_k(0) + E_p(0) = \frac{1}{2}mv^2(0) + mg \cdot 0 \overset{!}{=} E_k(H) + E_p(H) = \frac{1}{2}mv^2 + mgH.$$

Therefrom, $v(0) = \sqrt{v^2 + 2gH}$, regardless the angle of elevation.

13 Let's draw the paths of boats:



Denote $d_1 = 3d$, $d_2 = 4d$ and $d_3 = 21d$. We can clearly see that

- a boat A from the point K to the point L gets in time t and covers distance d_1. We can easily calculate its velocity $v = \frac{d_1}{t} = \frac{3d}{t}$.
- a boat D from the point L to the point M gets in time t too and covers distance $d_2 = 4d$. Its velocity is $u = \frac{d_2}{t} = \frac{4d}{t}$.
- a sailing of the boat A from the point L to the point N takes time $5t$ and a covered distance is $s_1 = v \cdot 5t = 5d_1 = 15d$.
- a sailing of the boat D from the point L to the point O takes time $6t$ and a covered distance is $s_2 = u \cdot 6t = 6d_2 = 24d$.

We are to find an angle α. We have found the lengths of all sides of a triangle $\triangle LON$, thus the sought angle can be determined using the low of cosines

$$d_3^2 = s_1^2 + s_2^2 - 2s_1 s_2 \cos\alpha,$$

from where

$$\alpha = \arccos\frac{s_1^2 + s_2^2 - d_3^2}{2s_1 s_2} = \arccos\frac{25d_1^2 + 36d_2^2 - d_3^2}{60d_1 d_2}.$$

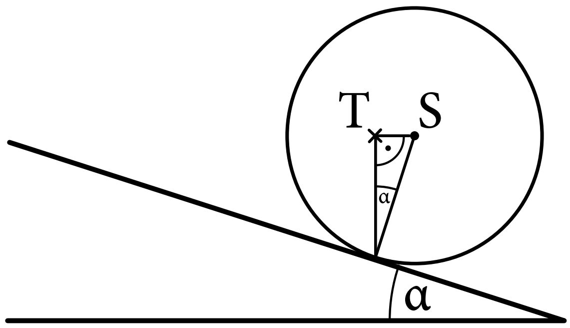
After substituting for d_1, d_2 and d_3 we obtain

$$\alpha = \arccos \frac{225d^2 + 576d^2 - 441d^2}{720d^2} = \arccos \frac{1}{2} = 60°.$$

14 If the sheep managed to climb the hill, a friction had to be sufficiently large, so that the sheep did not slip down. Therefore, in order the sheep to move, it has to be able to rotate around a point of contact with the hill. In order the sheep not to move, a torque calculated with respect to the point of cantact with the hill has to be non-negative, i. e. directed to the hill. The total torque is a result of three forces acting on the sheep:

- gravity acting in the centre of mass;
- normal force acting in the point of contact with the hill;
- frictional force acting also in the point of contact.

Forces acting in the point of contact do not contribute to the total torque, as a length of their moment arm is zero. The only force that has to be included into a calculation is gravity. In order the sheep to remain on the hill, the torque has to be zero. As the gravity always acts downwards, the centre of mass of the sheep has to be situated directly above the point of contact.



Maximal horizontal distance of a centre of the sheep and the point of contact, and hence the steepest slope corresponds to the case, in which the centre of mass of the sheep is in the same height as its geometric centre. An angle between the centre of mass, the point of contact and the geometric centre is equal to the slope of the hill α in this limit situation. Therefore

$$\sin \alpha = \frac{R/4}{R} = \frac{1}{4} \quad \Rightarrow \quad \alpha \le \arcsin \frac{1}{4} \doteq 14{,}48°.$$

15 Denote individual times as follows:

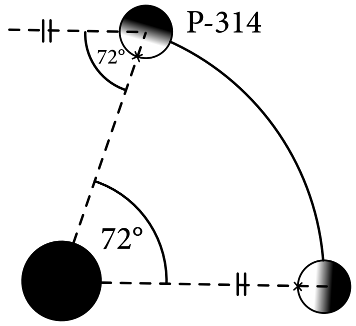
- time between two sunsets t_1;
- length of a year t_2;
- sought rotational period t.

The planet performs $\frac{t_1}{t}$ rotations and travels $\frac{t_1}{t_2}$ of its orbit between two consecutive sunsets. It corresponds to a state in which the planet has got a same orientation with respect to its sun again. A picture illustrate that the planet has performed one whole rotation and an additional fraction of another, which is exactly equal to the travelled fraction of orbit. In terms of math

$$\frac{t_1}{t} = 1 + \frac{t1}{t2}.$$

Therefrom

$$t = \frac{t_1 t_2}{t_2 + t_1} = 50\ \text{h}.$$



16 Denote the four important points of the necklace A, B, C and D.

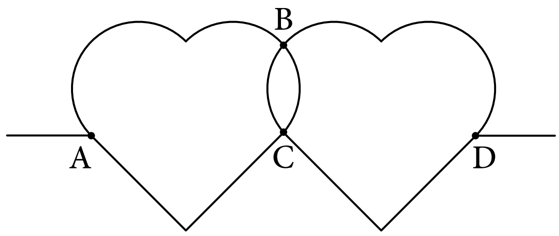


Figura 16.1: *Four important points of the necklace*

An effective resistance between points A and B is $\frac{3}{2}\pi r\rho$, and because of the symmetry, the resistance between B and D is same. Similarly, the resistance between the points A and C, as well as between C and D, is $4r\rho$.

Based on the symmetry, it can be also concluded that the potential in the points B and C is equal and we can get rid of the wire connecting them. Therefore, we get two parallel wires, one with the resistance of $8r\rho$ and the other with the resistance of $3\pi r\rho$.

The resistance of the entire circuit is therefore

$$\frac{1}{\dfrac{1}{8r\rho} + \dfrac{1}{3\pi r\rho}} = \frac{24\pi}{3\pi + 8} r\rho \doteq 4{,}327 r\rho.$$

17 Since voltage in the circuit is constant, the power output of the heater only depends on its resistance. Power can be expressed as the product of voltage and current; however, current flowing through the circuit is uniquely determined by the value of resistance as $I = U/R$. Hence

$$P = UI = \frac{U^2}{R}.$$

If we enumerate all possible ways how Luke can connect the resistors, we will find out there are 18 distinct schemes:

- no resistors (one scheme),

- a single resistor (three ways how to choose it),
- two resistors in series (three ways which to omit, order is not important),
- three resistors in series (only one way, order is not important),
- two resistors in parallel (three ways, which one is omitted),
- two resistors in parallel and the third one in series to them (three ways),
- two resistors in series and the third one parallel to them (three ways)
- and finally all three resistors in parallel (a single way).

In order not to destroy the fuse, the total resistance must be at least $\frac{230\ \text{V}}{15\ \text{A}} \doteq 15{,}5\ \Omega$. The scheme without resistors will short-circuit the heater and blow the fuse. The scheme with three resistors in parallel will produce total resistance only

$$R = \cfrac{1}{\cfrac{1}{20\ \Omega} + \cfrac{1}{30\ \Omega} + \cfrac{1}{60\ \Omega}} = 10\ \Omega,$$

which is too little and destroys the fuse as well. The same is true for resistors 20 a 30 Ω; and also 20 a 60 Ω when connected in parallel – their resistances are only 12 Ω and 15 Ω respectively. All other schemes are safe.

So we are left with 14 schemes. However, two of them have the same resistance: if we connect the second and third resistor in parallel, we obtain $R = 20\ \Omega$, which is equal to using only the first resistor. Therefore, there are only 13 possible power settings of Luke's heater.

18 As the hands move with constant average angular velocities and pass the XII mark simultaneously, there are certainly two similar solutions, differing only in the direction of the torque; and both are equally far away from the midnight or noon. We are going to analyse only the solution where the force acts anticlockwise, which occurs in the evening. The morning solution is its mirror image with respect to the vertical axis.

At first, we need to realise that the minute hand is twice as long as the hour hand, but its mass is only half as much, so both hands exert the same torque as long as the angles are equal. The time needed to complete one revolution is 43200 seconds for the hour hands and 3600 seconds for the minute hand. We may express the acting torques in terms of angles (measured from the XII mark)

$$M = M_{\text{hour}} + M_{\text{min}} = 2md \sin\left(\frac{2\pi t}{43\,200\ \text{s}}\right) + 2md \sin\left(\frac{2\pi t}{3600\ \text{s}}\right). \tag{18.1}$$

To find the maximum, we may differentiate this function. If we are not able or willing to do that, it is reasonably simple to think a bit and then use a calculator to find the solution by trial and error. Also values of m and d are constant, and since we only want to know the time and not the magnitude of the torque, we may ignore them altogether.

We should intuitively understand than the hands will exert maximum torque at about quarter to nine, when both hands are close to the IX mark. However, at 08:45 exactly the hour hand is still 3,75° below the mark and its torque is still increasing with time. The exact maximum will thus occur slightly later.

For the exact second 08:45:00 we substitute $t = 31\,500$ s to 18.1 and using a calculator we find out that

$$M_{08:45:00} \doteq 1{,}991\,445 \cdot 2mg.$$

By trial and error we do the same for next few seconds. Soon we will find out that the torque increases slightly a few more times, but after $t = 31\,506$ s = 08:45:06 it starts to decrease again. Other local maxima between 06:00 a 12:00 are necessarily smaller.

Finally we may want to find the mirrored solution, or the complement to 43200 seconds. So the solutions are

$$t = 11\,694 \text{ s} \equiv 03{:}14{:}54, \qquad \text{and} \qquad t = 31\,506 \text{ s} \equiv 08{:}45{:}06,$$

or, in both cases, exactly twelve hours later.

19 A cube immersed into a liquid follows the Archimedes' principle. Its straightforward consequence is that a ratio of densities of the cube and the liquid is equal to a ratio of an immersed volume of the cube to the volume of the entire cube. Let a be a length of cube side, ρ_w a density of water and ρ_c a density of the cube, then in the first situation,

$$\frac{\rho_c}{\rho_w} = 1 - \frac{a^2 \cdot 4\text{ cm}}{a^3}. \tag{19.1}$$

In the second situation, the a height of the emerged part is 18 cm. It has got a shape of a regular pyramide, the upper edges of which form right angles.

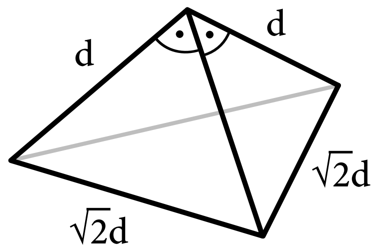


Figura 19.1: *An emerged part of the cube*

Denote a length of these edges d. A volume of the pyramide is $\frac{1}{6}d^3$. Based on the Pytagorean theorem, edges along a base of the pyramid are $\sqrt{2}d$ long. Let's focus on the triangular base of the pyramide.

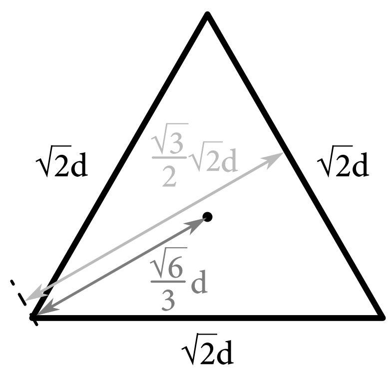


Figura 19.2: *The intersection of the cube and a water surface*

It has a shape of an equilateral triangle with an altitude $\frac{\sqrt{6}}{2}d$ long. A distance of a centre of the based from its vertices is $\frac{\sqrt{6}}{3}d$. Finally, consider a right-angled triangle determined by the centre of the base, a vertex of the base and the upper vertex of the pyramide.

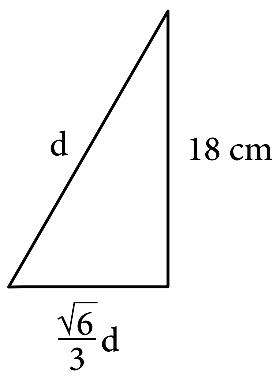


Figura 19.3: *The considered right-angled triangle*

The Pythagorean theorem gives $d = 18\sqrt{3}$ cm. In the second situation,

$$\frac{\rho_c}{\rho_w} = 1 - \frac{\sqrt{3} \cdot (18\ \mathrm{cm})^3}{2a^3}, \tag{19.2}$$

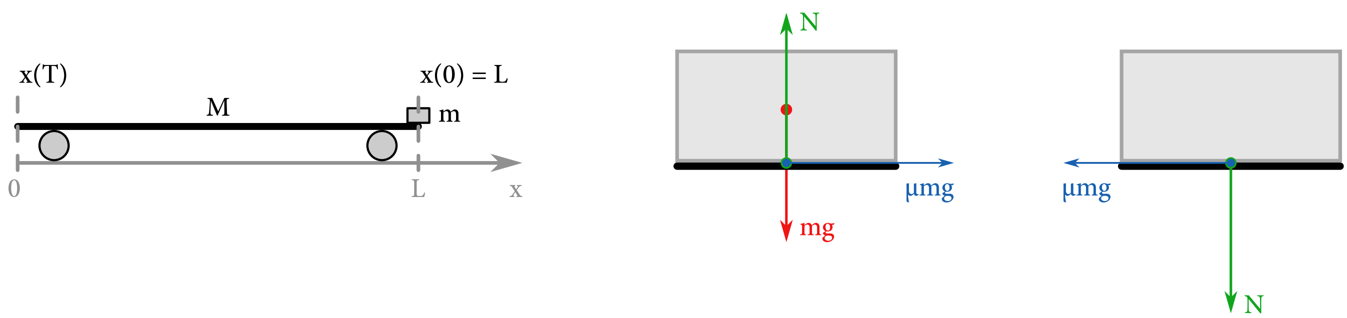
and comparing 19.1 and 19.2 yields

$$a = 3^{\frac{13}{4}}\ \mathrm{cm}.$$

Finally, substituting into 19.1 we get

$$\rho_c = \left(1 - 4 \cdot 3^{-\frac{13}{4}}\right) \rho_w \doteq 887\ \mathrm{kg/m^3}.$$

20 Shortly after accelerating the skateboard, the stone is still in rest at the front end of the skateboard. The skateboard accelerated to an initial velocity **v** starts to decelerate immediately due to a friction force between the stone and the skateboard. As a consequence of Newton's third law, the stone starts to accelerate. The situation is depicted on the picture:



Firstly, we find accelerations of the stone **a** and of the skateboard **A**:

$$a = \mu g, \qquad A = -\mu \frac{m}{M} g. \tag{20.1}$$

Then we can write equations for position and velocity of the stone with an initial position L

$$x_K(t) = L + \frac{1}{2}\mu g t^2, \tag{20.2}$$

$$v_K(t) = \mu g t \tag{20.3}$$

and of the skateboard with an initial position 0

$$x_S(t) = vt - \frac{1}{2}\mu g \frac{m}{M} t^2, \tag{20.4}$$

$$v_S(t) = v - \mu \frac{m}{M} g t. \tag{20.5}$$

In order the stone not to fall down of the skateboard, it has to stop moving with respect to the skateboard by it passes distance L. Denote the entire time since the start of moving to an equalisation of velocities as τ. We can state following conditions for position and velocity

$$x_S(\tau) = x_K(\tau). \tag{20.6}$$

$$v_S(\tau) = v_K(\tau), \tag{20.7}$$

After applying of 20.5 and 20.3, the condition for velocity 20.7 yields an equation for τ:

$$v - \mu \frac{m}{M} g\tau = \mu g \tau, \tag{20.8}$$

therefrom

$$\tau = \frac{v}{\mu g}\frac{M}{m+M}. \tag{20.9}$$

Analogiously, after substituting of 20.4 and 20.2 into the condition for position 20.6, we obtain

$$L + \frac{1}{2}\mu g \tau^2 = v\tau - \frac{1}{2}\mu g \frac{m}{M}\tau^2.$$

Substituting for τ from 20.9 leads to

$$L = v\tau - \frac{1}{2}\mu g \frac{m+M}{M}\tau^2,$$

$$L = v\frac{v}{\mu g}\frac{M}{M+m} - \frac{1}{2}\mu g \frac{m+M}{M}\frac{v^2}{(\mu g)^2}\left(\frac{M}{M+m}\right)^2,$$

$$L = \frac{v^2}{\mu g}\frac{M}{m+M} - \frac{1}{2}\frac{v^2}{\mu g}\frac{M}{m+M}.$$

Therefrom

$$v = \sqrt{2\mu g L\left(\frac{m+M}{M}\right)},$$

where we have omitted a negative solution as being non-physical.

21 Let's start with a simpler and more obvious condition: at the moment of landing when the height of a centre of mass above the ground is zero, the smartphone must be rotated by a multiple of 2π. The smartphone falls in homogeneous gravity field with gravity g from height h in time

$$t = \sqrt{\frac{2h}{g}}.$$

The smartphone on the table was laying on its back side, thus it has to be rotated by an angle $2\pi k$ in time t, where k is an arbitrary integer. Therefore

$$\omega = \frac{2\pi k}{\sqrt{\frac{2h}{g}}}.$$

The second relevant condition is that at the time of landing, upper and bottom edges of the smartphone must move downward. Otherwise, they would need to rise from a floor what is physically impossible and furthermore, it contradicts an assumption that landing is the first touch with the floor. A slower edge moves with velocity

$$v = -gt + \omega\frac{l}{2}\cos\varphi,$$

where φ is an angle between the smarthphone and the floor. At the moment of hitting the ground, it has to be 0°, thus a condition $v \leq 0$ leads to

$$\sqrt{2hg} = gt \geq \omega\frac{l}{2}.$$

We are interested in maximal allowed angular velocity ω of the smartphone, at which both conditions are still satisfied. Applying both conditions results in a condition

$$k \leq \frac{2h}{\pi l},$$

which yields the highest possible number of rotations $k = 3$ for given numeric values.

22 In order to secure that the beam returns along the same trajectory as it has entered, it has to be reflected perpendicularly from one of the the mirrors. Let's reverse the situation and trace the beam backwards, starting with the beam coming out of one of the mirrors at an angle of 90°. Firstly, let's assume that the beam will carry on reflecting from the mirrors forever. Later on, it will become clear why it is useful.

For the first time, the beam hits the mirror at an angle $90° - \alpha$. Of course, it will reflect at the same angle. The picture depicts a situation in which the beam hits one of the mirrors at an angle $90° - k\alpha$ after a few reflections:

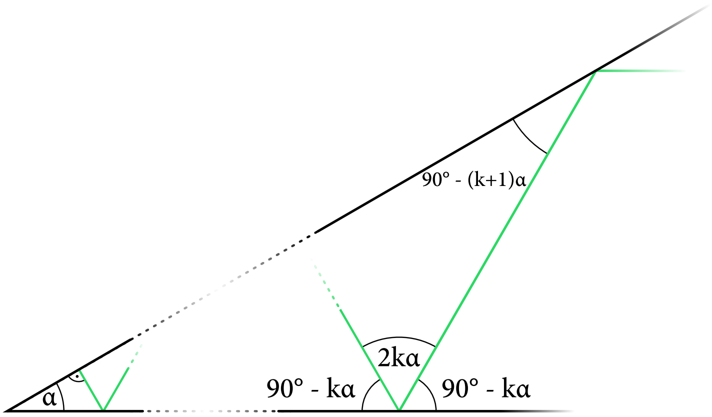


Figura 22.1: *The beam reflects until it reflects at an angle kα.*

The sum of angles in a triangle reveals that the beam wil hit the mirror at an angle $90° - (k+1)\,\alpha$ at the next reflection. Therefore, the beam reflects sequentially at the angles $90°, 90° - \alpha, 90° - 2\alpha, 90° - 3\alpha, \ldots$

The beam leaves the system in parallel to one of the mirrors only if one of those angles is equal to 0°. Therefore, we look for angles α, which satisfies the condition $90° - k\alpha = 0°$ for any natural number k. Hence, the solutions are all angles in the form $90°/k$.

23 First of all, we have to realise that bounces are purely elastic and hence the first bounce just turns the direction of the x-component of velocity of bouncy ball. Hence, we can imagine the wall as 'mirrors' as if the ball's motion is mirrored by the presence of walls. Let's imagine that there is no wall and the floor continues. Then, at the distance $2d$ from Charlie, there stands another Charlie who catches the ball right after it has bounced from the floor:

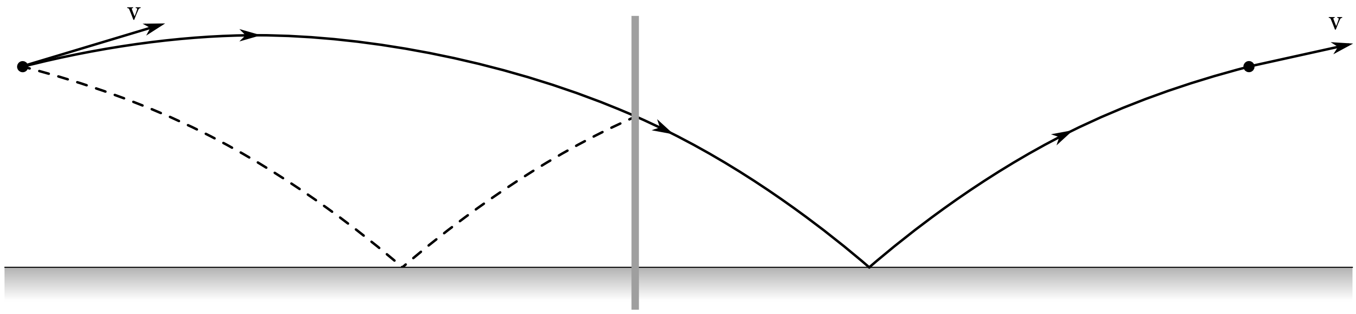


Figura 23.1: *Equivalent situation after 'mirroring'.*

Since the second bounce is also purely elastic, the y-component of velocity just turns its direction. Then, due to the law of conservation of energy, it flies to the very same height as at the start, where it has the very same velocity as at the start. This situation is path-equivalent to the case when the ball is thrown from ground, up to the fact that a piece of the parabolic path is cut from its start and put at its end. Therefore, the travel time (denoted by T) is same in both situations.

Let's consider bounce ball thrown at angle φ at velocity v_0 in a way that path covers distance $2d$ and at height h it has velocity v.

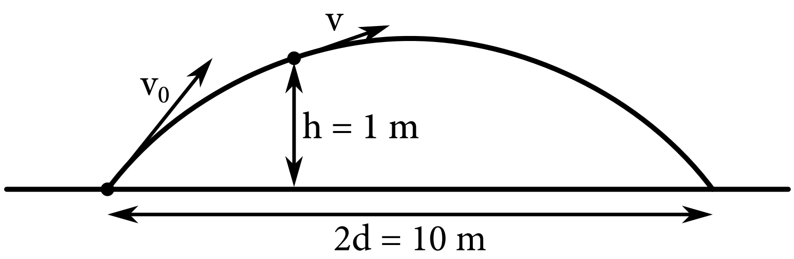


Figura 23.2: *Throw for which we want to calculate the angle.*

In order to calculate a time the ball needs to cover a parabolic path, we need its initial velocity v_0 with respective components v_x and v_y. This can be expressed from the law of conservation of energy,

$$v_0^2 = v_x^2 + v_y^2 = v^2 + 2gh. \tag{23.1}$$

A projectile motion is governed by equations

$$\begin{aligned} x(t) &= v_x t, \\ y(t) &= v_y t - \frac{1}{2}gt^2. \end{aligned} \tag{23.2}$$

We know initial and final coordinates

$$\begin{aligned} x(0) &= 0 \qquad \text{a} \qquad x(T) = 2d, \\ y(0) &= 0 \qquad \text{a} \qquad y(T) = 0. \end{aligned} \tag{23.3}$$

Having substituted from 23.2 to right-hand-sides, we obtain

$$T = \frac{2d}{v_x},$$

and excluding of one T from the second equation yields

$$v_y = \frac{gT}{2} = \frac{gd}{v_x}.$$

It can be substituted into 23.1 and we obtain

$$v_x^2 + \frac{g^2d^2}{v_x^2} = v_0^2.$$

Having multiplied by v_x^2, we obtain a biquadratic equation

$$v_x^4 - v_0^2v_x^2 + g^2d^2 = 0,$$

which has two solutions

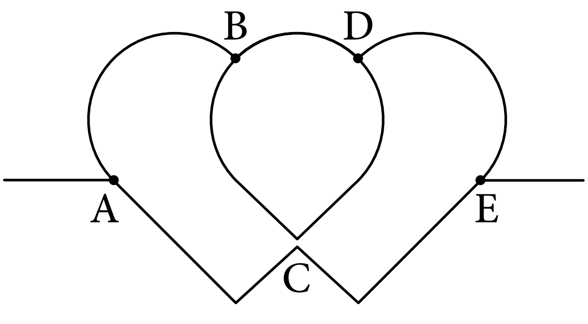
$$v_x^2 = \frac{v_0^2 \pm \sqrt{v_0^4 - 4g^2d^2}}{2}.$$

One of them does not fulfil all required conditions. If we threw the ball with this horizontal velocity, the vertical component of the velocity would be too small for the ball to reach the height h. The situation as described in the formulation of the problem could not occur. Therefore, the sought solution is the one with a minus sign. Thus, the resulting time is

$$T = \frac{2d}{v_x} = \frac{2d}{\sqrt{\frac{v^2+2gh-\sqrt{(v^2+2gh)^2-4g^2d^2}}{2}}}.$$

Having inserted a provided numerical values, we find that $T \doteq 2{,}425$ s, or, having used more precise value of g, it is slightly more.

24 Let's name the nodes A, B, C, D, E. Thanks to the symmetry, the current flowing from A to C (not through B), is same as the current from C to E. Similarly the current flowing from B to C is same as the current from C to D. Therefore, the circuit can be split into two separate branches at the point C:



The resistance between A and B, and as well between D and E, because of the symmetry, is

$$R_{AB} = R_{DE} = \pi r\rho.$$

The resistance between the points A and E is $6r\rho$. There are two parallel wires between the points B and D, one with the resistance of $\frac{1}{2}\pi r\rho$ and the second one with the resistance of $(2+\pi)\,r\rho$.

Therefore, the resulting resistance between the points B and D is

$$R_{BD} = \frac{2\pi+\pi^2}{3\pi+4} r\rho.$$

And then the resistance of the upper branch between the points A and E is

$$\left(2\pi + \frac{2\pi+\pi^2}{3\pi+4}\right) r\rho = \frac{7\pi^2+10\pi}{4+3\pi} r\rho$$

Therefore, the total resistance of the entire circuit is

$$\frac{42\pi^2+60\pi}{24+28\pi+7\pi^2} r\rho \doteq 3{,}3306 r\rho.$$

25 Denote height of a barrel H. The barrel is half-full, thus a volume of the air inside is $V_0 = \frac{H}{2}S$. A pressure in the barrel is equal to the atmospheric pressure. A pouring of the tea stops at the moment when a sum of the air pressure inside and a hydrostatic pressure of the tea is equal to the atmospheric pressure outside.

Therefore,

$$p_a = p + \left(\frac{H}{2} - h\right)\rho g,$$

where h is a descent of a tea surface and ρ is a density of the tea.

A temperature of the air inside after stopping pouring of the tea gradually reaches the temperature of the air before pouring. According to the ideal gas low

$$p_0\frac{H}{2}S = \left[p_a - \left(\frac{H}{2} - h\right)\rho g\right]\left(\frac{H}{2} + h\right)S.$$

Having rearanged the last equation, we obtain a quadratic equation for h

$$h^2 + \frac{p_a}{\rho g}h - \frac{H^2}{4} = 0.$$

Its positive solution is a sought descent of the tea surface

$$h = -\frac{p_a}{2\rho g} + \sqrt{\left(\frac{p_a}{2\rho g}\right)^2 + \frac{H^2}{4}}.$$

For $H = 1$ m, we obtain $h \doteq 25$ mm.

26 We will find an elastic energy of springs in the frame. An energy of a spring with stiffness k, elongated by x is

$$E = \frac{1}{2}kx^2.$$

Suppose, that an angle between two adjecent sides of the frame is α. Then one spring has length $x_1 = 2d\sin\frac{\alpha}{2}$ and the other one length $x_2 = 2d\cos\frac{\alpha}{2}$. Therefore, the overal energy of the frame is

$$E = 2kd^2\left(\sin^2\frac{\alpha}{2} + \cos^2\frac{\alpha}{2}\right) = 2kd^2.$$

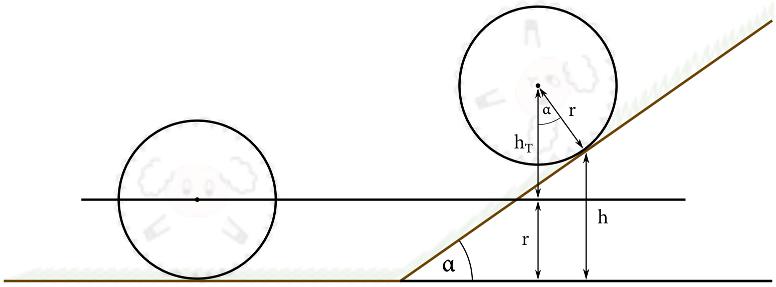
We see that the energy is independent of the angle α. It means that the force, which is necessary to be exerted in order to maintain the angle α, is zero.

27 We solve this problems utilising the law of conservation of energy. The sheep has got only a potential energy at the highest point of its path – at height h above flat ground. The potential energy is mgh_T, where m is sheep's mass and g is gravity. The rolling sheep has got both translational and rotational kinetic energy. An angular frequency of the rolling sheep is v/r and a moment of inertia of a homogeneous cylinder is $\frac{1}{2}mr^2$. Therefore, the total kinetic energy of the sheep is

$$\frac{1}{2}mv^2 + \frac{1}{4}\frac{mr^2v^2}{r^2} = \frac{3}{4}mv^2.$$

From the law of conservation of energy, we obtain

$$h_T = \frac{3v^2}{4g}.$$



Notice that the sheep does not touch the hill with its lowest point but with bit higher. Namely, its lowest point is r below its axis and a point of contact is $r\cos\alpha$ below the axis. Therefore, the grass on the hill is rolled over upto height

$$h = \frac{3v^2}{4g} + r\left(1 - \cos\alpha\right).$$

28 Based on the symmetry, we can conclude that each blade (i. e. a triangular part) contributes to total moment of inertia by the same portion. Therefore, we need to express a moment of inertia just for one blade and multiply it by three. Besides that, we can notice that a square with side a can be made from two blades.

We are interested in the moment of inertia with respect to the perpendicular axis, which goes through one of the apexes. The moment of inertia with respect to the perpendicular axis going through the center can be found in tables,

$$I_\square = \frac{1}{6} m_\square a^2.$$

If we do not have the tables, we can calculate it by using scaling and Steiner's theorem. It allows us to calculate the moment of inertia around the axis going through the apex.

If we denote the distance of new axis from the center of gravity $x = \frac{\sqrt{2}}{2}a$, we will obtain

$$I_{\square'} = I_\square + m_\square x^2 = \frac{1}{6} m_\square a^2 + m_\square \left(\frac{\sqrt{2}}{2} a^2\right) = \frac{2}{3} m_\square a^2.$$

Stella's propeller consists of three halves of stated square, therefore its moment of inertia is 3/2 times bigger, thus $m_\square a^2$. Having substituted $m_\square = \sigma a^2$, we obtain

$$I_{\text{propeller}} = \sigma a^4.$$

We can solve the problem the other way around. We just need to realise that moment of inertia depends only on the perpendicular distance of points of the body from the rotational axis. The distance does not change if we arbitrarily rotate the blades of the propeller around the rotation axis. It means that we can rotate them in a such way that we get a shape, which represents 3/8 of square with a side of legth $2a$. The propeller's moment of inertia is then 3/8 of the moment of inertia of the big square around its center, thus

$$I_{\text{propeller}} = \frac{3}{8} \cdot \frac{1}{6} \cdot (2a)^2 \, \sigma \cdot (2a)^2 = \sigma a^4.$$

Moment of inertia is usually given in form constant times mass times characteristic length squared. Mass of the whole propeller is $M = \frac{3}{2}\sigma a^2$. If we use the length of the triangle's leg a as a characterictic length, the moment of inertia will be

$$I_{\text{propeller}} = \frac{2}{3}Ma^2.$$

29 Imagine a spring composed of n identical springs with stiffness k connected in series. If we exert a force F on the composed spring, a relative shortening of each spring is same as the relative shortening of the composed spring. An absolute shortening is n-times greater than a shortening of the individual springs and a total stiffness of the composed spring is only $\frac{k}{n}$.

In case of cutting the spring, it works reversely – if we cut the spring into n pieces, and exert a force F, an absolute shortening of one piece is n-times less than a shortening of the original spring, what corresponds to a stiffness nk.

If we connect n identical springs and we want to compress them by a certain length, we have to exert n-times greater force comparing to compresing only a single spring. It means that a total stiffness of the parallel connection of n identical springs with the stiffness k is nk. Therefore, the total stiffness of spring's pieces is kn^2 if they are placed next to each other.

We have to realise that the rest length of those springs also decreased to a value $\frac{1\,\mathrm{m}}{n}$. In order to hold the wooden platform above the ground, the springs must be compressed by less than their rest length. A force due to the wooden platform with Horacio exerted on the springs is equal $F_S = g \cdot 110\,\mathrm{kg}$.

A force due to the spring is $F_p = k'x = kn^2\Delta x$, where Δx is change in length of the spring. The condition for holding the wooden platform above the ground is

$$F_S < n \cdot 70\,\mathrm{N}.$$

It is easy to finish the calculations and obtain that Horacio has to cut the spring into at least 16 pieces.

30 We are to calculate an illuminated solid angle. It is equal to an area of a sphere which would be illuminated if we placed the cube to the centre of the sphere, divided by squared radius of the sphere. The easiest way how to approach the problem is to consider a sphere with a centre in the centre of the cube and with a diameter equal to the length of a face diagonal of the cube.

Denote the length of edges of the cube a, then a radius of the sphere is $\frac{\sqrt{2}}{2}a$. Such a sphere just fits the holes in the faces of the cube. If we scaled down the cubed, protruding parts of the sphere beyond the vertical faces[2] of the cube would remain illuminated.

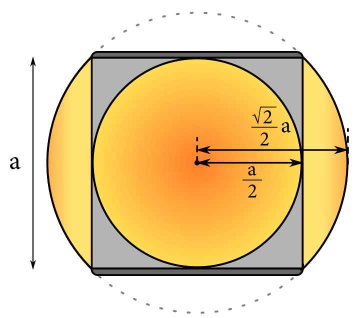


Figura 30.1: *An illuminated solid angle as a part of a spherical surface*

[2] holes are only in the vertical faces

In order to determine the solid angle of the illuminated parts of the sphere, we have to find their area. These parts are spherical caps, the area of which is $2\pi rh$, where r is a radius of the sphere and h is a height of the spherical cap. Based on symmetry, all four caps have got the same area. Their height is a difference between the radius of the sphere and a half of the length of the cube edge, hence

$$\frac{\sqrt{2}}{2}a - \frac{a}{2}.$$

Each of the caps has got the area of

$$2\pi\left(\frac{\sqrt{2}}{2} - \frac{1}{2}\right)\frac{\sqrt{2}}{2}a^2,$$

thus a total area of illuminated parts of the sphere is

$$8\pi\left(\frac{\sqrt{2}}{2} - \frac{1}{2}\right)\frac{\sqrt{2}}{2}a^2.$$

Having divided it by the squared radius of the sphere, we obtain the sought solid angle

$$8\pi\left(1 - \frac{\sqrt{2}}{2}\right) \doteq 7{,}361 \text{ sr}.$$

31 A state of the gas is described by so-called state quantities. Commonly used state quantities are pressure p, volume V and temperature T. State variables has to satisfy an equation of state

$$pV = Nk_BT,$$

where N is number of gas particles and k_B is Boltzmann constant. It means that if we know two state quantities, the third one can be calculated.

If only forces that can act on piston are a force due to a spring and a force due to a pressure inside, the pressure inside satisfies $p = \frac{kh}{S}$, where k is a stiffness of the spring, h is its elongation and S is an area of the piston. It means that the elongation of the spring uniqly describes gas pressure, therefore it is also a good state quantity, as well as it describes a volume of the gas $V = Sh$. Inserting these expresions into the state equation yields

$$kh^2 = Nk_BT.$$

Realise that the elongation of the spring describes both a pressure and a volume. It means that if we know a volume, we can immediately calculate a pressure (and vice versa), therefore there is only one independent state quantity.

Oliver drew an h–T diagram of an adiabatic process with an ideal gas. An adiapatic process follows an equation

$$pV^\varkappa = \text{const.}$$

In our case, we obtain

$$kS^{\varkappa-1}h^{\varkappa+1} = \text{const.}$$

It means that if we deal with the adiabatic process, $h = \text{const.}$ However, if there are no external forces acting on the system, the elongation of the spring will not alter. Thus, neither the pressure nor the volume will alter, and hence neither the temperature can alter. It means that if we are to consider the adiabatic process, e. i. if we forbid a heat transfer with surrounding, the gas state will not alter, therefore the adiabatic process will be represented by a single point on the h–T diagram.

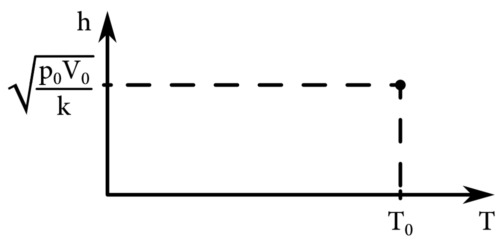
Let's find coordinates of this point. The state of the gas is given by state quantities p_0, V_0, T_0. Hence, the temerature is given, thus we need to find only the elongation of the spring h_0. Multiplying equations for pressure and for volume yields

$$p_0 V_0 = \frac{kh_0}{S} \cdot Sh_0 = kh_0^2,$$

therefrom

$$h_0 = \sqrt{\frac{p_0 V_0}{k}}.$$

Therefore, the sought diagram of the adiabatic process looks like this:



32 We will consider light being a transverse electromagnetic wave. Suppose, there is a light polarised in a certain direction. We let it pass through an ideal polarising filter with a plane of polarisation rotated with respect the direction of polarisation of the light by an angle φ. Behind the filter, we observe only a component in the direction determined by orientation of the plane of polarisation of the filter with an amplitude $A_0 \cos\varphi$. A perpendicular component with a magnitude $A_0 \sin\varphi$ does not pass through the filter.

Energy of an electromagnetic wave is equal to a squared amplitude, hence in terms of intensities, a portion with a magnitude $I_0 \cos^2\varphi$ passes, whilst a portion with a magnitude $I_0 \sin^2\varphi$ is absorbed by the filter. A sum of transmitted and absorbed energy is, of course, equal to the energy of entering wave.

If Simon wants horizontaly polarised light, the last filter has to be oriented horizontaly. What about other filters? Intuitively, we guess that they should be oriented in such a way that angles between planes of polarisation of two neighbouring filters were same and as little as possible. Let's prove it.

Consider three filters with orientations described by angles φ_1, φ_2 and φ_3. Differences between angles of their orientation differs from each other,

$$\beta \equiv \varphi_3 - \varphi_2 \neq \varphi_2 - \varphi_1 \equiv \alpha.$$

We will show that if we rotate the middle filter in such a way that the differences are equal, a resulting intensity will increase. If we rotate only the middle filter, we have

$$\alpha + \beta = \gamma = \text{const} \quad \Rightarrow \quad \beta = \gamma - \alpha$$

thus the resulting intensity after passing the filters is

$$I_{\text{behind}} = I_{\text{in front}} \cdot \cos^2\alpha \cdot \cos^2(\gamma - \alpha).$$

By means of differential calculus or by a different method, we find out that this function reaches its maximum value when $\alpha = \gamma/2$, respectively $\alpha = \beta$. Rotating of the middle filter does not affect the direction of polarisation behind the third filter, therefore always, when we find such a triple of filters, we can optimese it by rotation of the middle filter. The only layout that cannot be optimesed is the one in which the differences between angles of orientations of the neighbouring filters are equal.

It also proves that Simon has to use all filters. If he ommited one of them, it would be same as if he placed it in front of the first one with same orientation. However, we already know that such a layout could be optimised by rotating of the second filter, therefore this layout cannot be optimal.

Finally, we need to calculate how much light passes through the optimal layout of the filters. Each of n filters rotates the plane of polarisation by angle $90°/n$ and lets pass only $\cos^2(90°/n)$ of intensity of the light. Each filter lowers the intensity by same factor, therefore the resulting intensity for $n = 10$ is

$$I = I_0 \cos^{20}\left(\frac{90°}{10}\right) \doteq 0{,}78\, I_0.$$

33 An explosion of a black powder supplied the gas with an energy E. Provided the explosion was sufficiently short, so that a projectile had no time to move, the gas did not do any work. Therefore all the supplied heat resulted to an increase of an internal energy of the gas. It lead to an increase of a temperature of the gas given by an equation $E = \gamma N k \Delta T$, wher N is a number of particles of the gas, k is Boltzmann constant and γ is a constant reflecting a number of degrees of freedom of gas molecules. For two-atomic molecules $\gamma = \frac{5}{2}$.

If an original temperature was T_0, a new temperature after the explosion was $T_1 = T_0 + \Delta T$. The explosion was an isochoric process, therefore a pressure after the explosion was

$$p_1 = p_0\left(1 + \frac{\Delta T}{T_0}\right).$$

Consequently, an aiabatic expansion occured. For adiabatic process, we have $pV^\varkappa = \text{const.}$, therefore $p_1 V_0^\varkappa = pV^\varkappa$. It yields

$$p(V) = p_0\left(1 + \frac{\Delta T}{T_0}\right)\left(\frac{V_0}{V}\right)^\varkappa.$$

The work done by the gas can be calculated by integrating

$$W = \int_{V_0}^{LS} p_0\left(1 + \frac{\Delta T}{T_0}\right)\left(\frac{V_0}{V}\right)^\varkappa \mathrm{d}V.$$

However, we can avoid evaluating the integral if we realise that there is no heat flow between the gas and its surrounding during an adiabatic process. Therefore, according to the first low of thermodynamics, a work done by the gas is equal to a negative change of a gas internal energy $W = -\delta U$. In order to determine a change of internal energy, we nee to find a change of temperature of the gas during the adiabatic expansion. According to the ideal gas law

$$p_2 = p_1 \frac{T_2}{T_0 + \Delta T}\frac{V_0}{LS}.$$

Having inserted into the equation of adiabatic process, we obtain

$$p_1 V_0^\varkappa = p_1 \frac{T_2}{T_0 + \Delta T}\frac{V_0}{LS}(LS)^\varkappa,$$

from where

$$T_2 = \left(\frac{V_0}{LS}\right)^{\varkappa-1}(T_0 + \Delta T).$$

Consequently, the sought change of internal energy is

$$\delta U = \gamma Nk\delta T = \gamma NK(T_0 + \Delta T)\left[\left(\frac{V_0}{LS}\right)^{\varkappa-1} - 1\right].$$

As $W = -\delta U$, we derive

$$W = (\gamma p_0 V_0 + E)\left[1 - \left(\frac{V_0}{LS}\right)^{\varkappa-1}\right].$$

The work done by gas is used for accelerating of the projectile with mass m. The projectile is provided with a kinetic energy $T = \frac{1}{2}mv^2$. Thus the sought velocity of the projectile is

$$v = \sqrt{\frac{2}{m}\left(\frac{5}{2}p_0 V_0 + E\right)\left[1 - \left(\frac{V_0}{LS}\right)^{2/5}\right]},$$

where we have used that $\gamma = \frac{5}{2}$ and $\varkappa = \frac{7}{5}$ for two-atomic gas.

34 Since we are only interested in luminous flux as viewed from a great distance and in the direction perpendicular to the shoreline, we only need to consider rays that exit the lanterns very close to this perpendicular line. Let's analyse a ray, exiting the lantern at a small angle α with respect to this line, for both lanterns.

With Nina's lantern, it is easy – the ray simply travels in a straight line forever. With Hannah's lantern, the ray also exits the lantern with angle α, but once it exits the glass cube, the angle will be different. We may determine this angle using Snell's law

$$n_1 \sin\alpha_1 = n_2 \sin\alpha_2,$$

where n_1 and n_2 are refractive indices of glass and air and α_1 and α_2 are angles at which the ray travel in and out of the cube.

Since angle α is very small, we may use the approximation $\sin\alpha \approx \alpha$. The refractive index of air n_2 is very close to 1, and we know that $n_1 = n$. Hence $\alpha_2 \approx n\alpha$.

All light exiting Hannah's lantern with angles less than α from the sailor's line of sight forms a cone with apex angle 2α. After refraction at the face of the glass cube, the apex angle of the new cone will be $2n\alpha$. The total luminous power stays constant, so the power density is inversely proportional to the area of the base of the cone. This is n^2 times larger for Hannah's light source, which mean the sailors will perceive Nina's lantern as n^2 times brighter.

35 A force acting on a moving charged particle with a velocity $\mathbf{v}$ and a charge q in a magnetic field with strength $\mathbf{B}$ is

$$\mathbf{F} = q\mathbf{v} \times \mathbf{B}.$$

In a considered case, the velocity of both particles is perpendicular to the magnetic field. Therefore, the force acting on the particles is a centripetal force. It can be expressed as

$$F_d = \frac{mv^2}{r},$$

thus

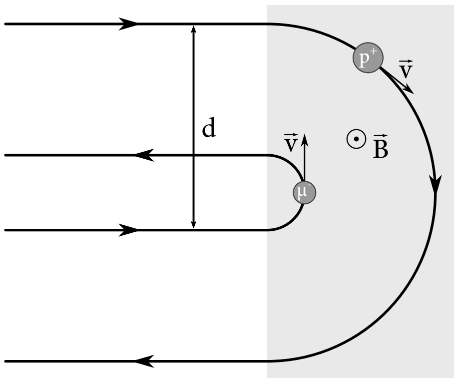
$$r = \frac{mv}{qB}.$$

Let's note that both particles move with relativistic velocities, therefore $m = \gamma m_0$, where m_0 is a rest mass of the particle and $\gamma = 1/\sqrt{1-\frac{v^2}{c^2}}$.

A muon charge differs from a proton charge only by a sign. In order to follow concentric half-circles inside the magnetic field, the particles had to entered the chamber at the mutual distance

$$d = r_e + r_p = \frac{v\left(m_p + m_\mu\right)}{eB\sqrt{1-\frac{v^2}{c^2}}}.$$

For given numeric values $d = 2{,}44$ m.



36 Suppose that blankets have reached thermal equilibrium, thus an absorbed power is radiated back to the surrounding space entirely. Based on the symetry, we can conclude that energy is radiated equally to both sides of the blankets. Considering, the blankets are sufficiently close to each other, the entire power radiated by one blanket towards other one is completely absorbed by the other blanket.

Enumerate the blankets starting with the blanket closest to the Sun. Areal power densities F[3] of individual blankets have to satisfy following set of equations

$$F_1 = F_\odot + \frac{1}{2}F_2, \qquad F_2 = \frac{1}{2}F_1 + \frac{1}{2}F_3, \qquad F_3 = \frac{1}{2}F_2.$$

The system of equations has following solution:

$$F_1 = \frac{3}{2}F_\odot, \qquad F_2 = F_\odot, \qquad F_3 = \frac{1}{2}F_\odot.$$

We are interested only in the last equation.

[3] F is the overal areal power density radiated by both sides of a blanket

Realise that we need to calculate an areal power density related to a unit surface area and it is only a half of F_3, as the blanket radiates only a half of its power in each direction. According to the problem statement, the blankets can be considered being ideal black bodies, therefore a temperature of the third blanket is

$$T = \sqrt[4]{\frac{F_\odot}{4\sigma}} \doteq 278{,}785\ \mathrm{K}.$$

37 Hydrostatic pressure increases with heigh as $p(h) = h\rho g$. A pressure in a liquid in a given point is independent of the orientation of surface, thus a magnitude of the force acting on a barrel at a height h is

$$\Delta F(h) = p(h) \cdot \Delta S = h\rho g \cdot 2\pi R \Delta h.$$

This force acts along the entire circumference of the barrel at the heigh h in the perpendicular direction to the barrel surface.

A tension force due to that force in a stripe of the barrel with thickness Δh at the height h can be determined based on the principle of virtual work. Imagine that the force due to the pressure extends the radius of the barrel by δR. The corresponding virtual work done on the barrel is $\delta W = \Delta F \cdot \delta R$.

Exactly the same virtual work would be done by the tension force if it caused the same deformation. As the circumference is lengthened by $2\pi\delta R$, the virtual work done by tension force is $\delta W = \Delta T \cdot 2\pi\delta R$. An equality of virtual works yields $\Delta T(h) = \frac{\Delta F(h)}{2\pi}$. Therefore, the tension force at height h is

$$\Delta T(h) = R\rho g h \Delta h.$$

The tension force stretches the stripe of thickness Δh. However, we are interested in the force which stretches the entire barrel, thus we need to sum a contribution of all stripes over the entire height of the barrel. One option is to evaluate the integral

$$T = \int_0^H R\rho g h \,\mathrm{d}h.$$

The other one is to realise that the increment of the tension force depends linearly on the height, therefore it is represented by a streight line crossing the origin on the graph. The overal tension force is the area below the line,[4] what is nothing more than the area of a right-angled triangle with catheti of lengths H and $R\rho g H$, thus the overal tension force is

$$T = \frac{1}{2} R\rho g H^2.$$

One rim can withstand the tension force given by its ultimate tensile strength and its crossectional area $\tau = \sigma S$, therefore the minimum number of rims is

$$N = \left\lceil \frac{R\rho g H^2}{2\sigma S} \right\rceil.$$

For the given numerical values, we obtain $N = 11$.

38 Let's build on a solution of the previous problem about polarising filters. The only difference is that an intensity decreases not only due to different orientations of the filters but also due to lower translucency of

[4] Similarly to uniformly accelerated motion, where the increment of distance is equal to the area below the graph $v(t)$.

the filters. We have to consider two counteracting effects – the more filters are used, the more economically the plane of polarisation can be rotated, but the higher are losses due to an absorption.

As a result, an additional exponential term appears in the formula for resulting intensity reflecting the loss of intensity when passing through each of the filters. We obtain

$$I = I_0 \cdot 0{,}9^n \cdot \left(\cos\frac{\pi}{2n}\right)^{2n}.$$

We examine a few first options (or alternativelly, we can use methods of differential calculus, what we do not recommend) and we find out that maximum is reached at $n = 5$,

$$I = I_0 \cdot 0{,}9^5 \cdot \cos\left(\frac{\pi}{10}\right)^{10} \doteq 0{,}36\ I_0.$$

39 Let's consider a small area of a soup bubble ΔS. We will start with a calculation of a force acting on this area due to a surface tension. The surface tension of the bubble results in a Laplace pressure p_c. The Laplace pressure is described by Young-Laplace equation. The bubble has two surfaces, thus

$$p_c = \frac{4\gamma}{R},$$

where γ is a surface tension of soup watter and R is a radius of the bubble. Then a force acting on the area ΔS is

$$\Delta F_k = p_k \Delta S = \frac{4\gamma}{R}\Delta S.$$

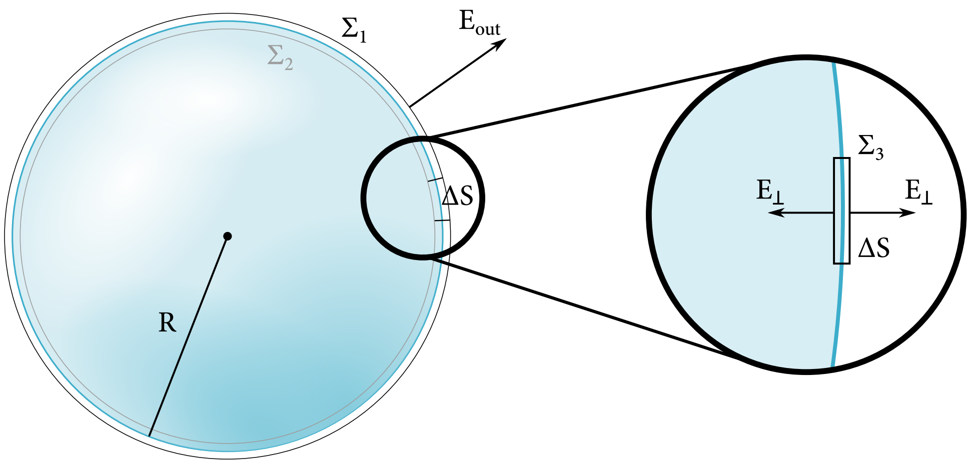
This force attracts the area to the bubble.

Secondly, we need to determine an electrostatic force which repels the area from the bubble after an electric charge has been brought onto the bubble. Assume that Mary has brought a charge Q. The bubble is conductive, therefore the charge has distributed uniformly over the entire outer surface of the bubble. An areal charge density on the bubble surface is

$$\sigma = \frac{Q}{4\pi R^2}.$$

A key tool for solving this problem is Gauss's law. It states that an integral of an electric intensity over an arbitrary closed surface is proportional to an overal electric charge inside the surface. In terms of math

$$\oint_S \mathbf{E} \cdot \mathrm{d}\mathbf{S} = \frac{q}{\varepsilon_0}.$$



A charge distribution on the bubble has got a spherical symmetry, hence the electric intensity outside the bubble has a spherical symmetry too. Consider a sphere Σ_1 at the outer surface of the bubble as a Gauss's surface. The intensity over this surface has got a radial direction and a constant magnitude. Therefore

$$\oint_{\Sigma_1} \mathbf{E}\cdot \mathrm{d}\mathbf{\Sigma_1} = E_{\text{out}} 4\pi R^2 \stackrel{!}{=} \frac{Q}{\varepsilon_0}.$$

Therefrom, the intensity at the bubble surface is

$$E_{\text{out}} = \frac{Q}{4\pi\varepsilon_0 R^2}.$$

Next, consider a sphere Σ_2 at the inner surface of the bubble as a Gauss's surface. In such a case, an overal charge inside the surface is zero and thus, based on the same arguments as in the previous situation, the electric intensity is

$$E_{\text{in}} = 0.$$

We have obtained all of this when we were looking at the bubble on a global scale. Now, let's look locally only at an area ΔS. If we are sufficiently close, we do not see a curvature of the bubble and the area seems to be flat. [5] Choose a Gauss's surface in a shape of a can Σ_3 with bases ΔS and with a tiny height. The can envelops the area ΔS of the bubble. As a charge distribution is planar, the electric intensity has got a reflection symmetry. The intensity has to be perpendicular to the area ΔS. If it were not, i. e. if it had also a tangential component with the area, it would cause an electric current flowing until the tangential component would be nullified by currents. According to Gauss's law

$$\oint_{\Sigma_3} \mathbf{E}\cdot \mathrm{d}\mathbf{\Sigma_3} = 2E_{\perp}\Delta S \stackrel{!}{=} \frac{\sigma\Delta S}{\varepsilon_0},$$

from where

$$E_{\perp} = \frac{\sigma}{2\varepsilon_0} = \frac{Q}{8\pi\varepsilon_0 R^2}.$$

The intensity due to the area ΔS is non-zero at the inner surface of the bubble. However, a total intensity inside the bubble is zero, thus an intensity due to rest of the bubble at the inner surface of the area ΔS has to

[5] Exactly due to a same reason why the Flat Earthers believe that the Earth is flat.

be $\mathbf{E} = -\mathbf{E}_\perp$. Consequently, a force repelling the area ΔS from the bubble is

$$\Delta F_e = E\Delta Q = e\sigma\Delta S = \frac{Q^2}{32\pi^2\varepsilon_0 R^4}\Delta S.$$

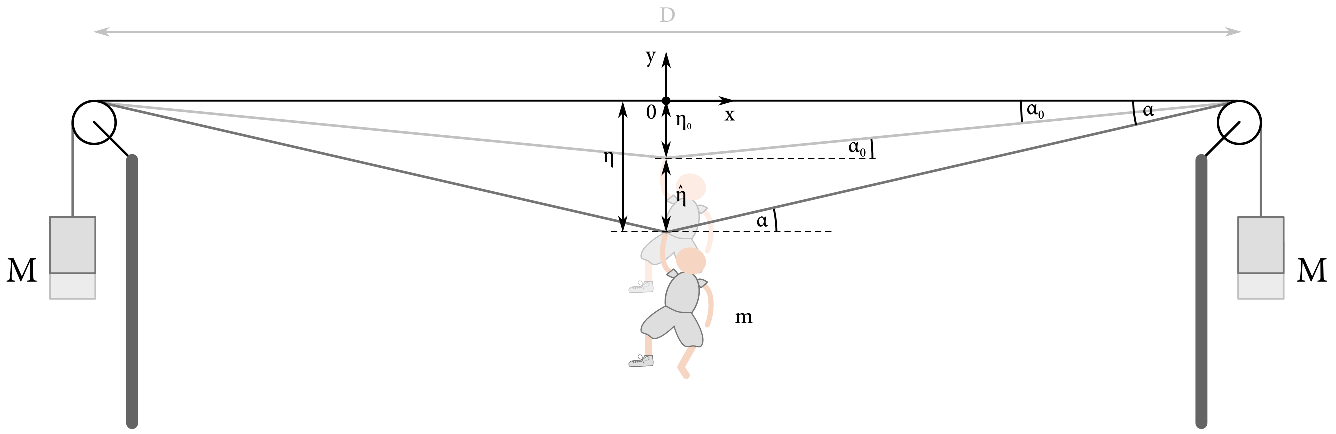
Finally, we can resolve the problem. If the pressure inside the bubble is to be equal to the atmospheric pressure, the attraction due to the surface tension has to be compensated by the electrostatic repulsion $\Delta F_c \stackrel{!}{=} \Delta F_e$. The stated equality yields

$$Q = \sqrt{128\pi^2\varepsilon_0\gamma R^3}.$$

For given numerical values, we obtain $Q \doteq 1{,}31 \times 10^{-7}$ C.

40 Let's start with determination of a depth of the sag of the rope in equilibrium. Denote the depth of the rope sag as η_0. Let α be an inclination of the line. Then

$$\sin\alpha_0 = \frac{\eta_0}{\sqrt{\frac{D^2}{4} + \eta_0^2}}.$$



The rope is stretched by a tensile force $T = Mg$. A balance of forces in a vertical direction yields

$$2Mg\sin\alpha_0 = mg.$$

Having substituted for $\sin\alpha_0$, we obtain

$$\frac{\eta_0}{\sqrt{\frac{D^2}{4} + \eta_0^2}} = \frac{m}{2M}.$$

For future use, we express the last equation in the following forms:

$$\eta_0 = \frac{mD}{2\sqrt{4M^2 - m^2}};$$

$$\sqrt{\frac{D^2}{4} + \eta_0^2} = \frac{MD}{\sqrt{4M^2 - m^2}}.$$

Now, let η denote an actual sag of the rope in a dynamical case. Introduce Cartesian coordinate system, as illustrated on the picture. Assume, that a length of the entire rope is L. In that case, we can find position vectors of individual objects. A position vector of the left mass is

$$\mathbf{r_L} = \left(-\frac{D}{2}; \sqrt{\frac{D^2}{4} + \eta^2} - \frac{L}{2}\right),$$

of the right mass

$$\mathbf{r_R} = \left(+\frac{D}{2}; \sqrt{\frac{D^2}{4} + \eta^2} - \frac{L}{2}\right)$$

and of Jacob

$$\mathbf{r} = (0; -\eta)\,.$$

Next, we find velocities of individual objects. Denote vertical component of Jacob's velocity as $\dot{\eta}$. Then, vertical components of velocities of masses are $\dot{\eta}\sin\alpha$, which can be obtained from the geometry. Therefore, the sought velocities are

$$\begin{aligned}
\mathbf{v_L} &= \left(0; \frac{\eta\dot{\eta}}{\sqrt{\frac{D^2}{4} + \eta^2}}\right); \\
\mathbf{v_P} &= \left(0; \frac{\eta\dot{\eta}}{\sqrt{\frac{D^2}{4} + \eta^2}}\right); \\
\mathbf{v} &= (0; -\dot{\eta})\,.
\end{aligned}$$

Finally, we are able to determine an overal potential energy of the system

$$U = Mgy_L + Mgy_R + mgy = 2Mg\left(\sqrt{\frac{D^2}{4} + \eta^2} - \frac{L}{2}\right) - mg\eta$$

and its kinetic energy

$$T = \frac{1}{2}Mv_L^2 + \frac{1}{2}Mv_R^2 + \frac{1}{2}mv^2 = \frac{1}{2}\left(\frac{2M\eta^2}{\frac{D^2}{4} + \eta^2} + m\right)\dot{\eta}^2.$$

We are interested in small oscillations of Jacob around his equilibrium state. Hence, express the sag of the rope in a form

$$\eta = \eta_0 + \hat{\eta},$$

where $\hat{\eta}$ denotes an amplitude of the small oscillations.

In case of the small oscillations, an expression for the potential energy can be expanded to Taylor series. We want to investigate harmonic oscillations, therefore we need an expansion upto the second order.

Let's find an expansion of a function

$$f(\hat{\eta}) = \sqrt{\frac{D^2}{4} + \left(\eta_0 + \hat{\eta}\right)^2}$$

around zero. We need to know a value of the function $f(\hat{\eta})$ and of its first two derivatives for that. Gradually, we obtain

$$f(\hat{\eta})\big|_{\hat{\eta}=0} = \sqrt{\frac{D^2}{4} + \eta_0^2};$$

$$\left.\frac{\mathrm{d}f(\hat{\eta})}{\mathrm{d}\hat{\eta}}\right|_{\hat{\eta}=0} = \left.\frac{\eta_0 + \hat{\eta}}{\sqrt{\frac{D^2}{4} + \left(\eta_0 + \hat{\eta}\right)^2}}\right|_{\hat{\eta}=0} = \frac{\eta_0}{\sqrt{\frac{D^2}{4} + \eta_0^2}};$$

$$\left.\frac{\mathrm{d}^2 f(\hat{\eta})}{\mathrm{d}\hat{\eta}^2}\right|_{\hat{\eta}=0} = \left.\frac{D^2}{4\sqrt{\frac{D^2}{4} + \left(\eta_0 + \hat{\eta}\right)^2}^3}\right|_{\hat{\eta}=0} = \frac{D^2}{4\sqrt{\frac{D^2}{4} + \eta_0^2}^3}.$$

Therefore,

$$\begin{aligned}
f(\hat{\eta}) &\approx f(\hat{\eta})\big|_{\hat{\eta}=0} + \left.\frac{\mathrm{d}f(\hat{\eta})}{\mathrm{d}\hat{\eta}}\right|_{\hat{\eta}=0} \cdot \hat{\eta} + \frac{1}{2}\left.\frac{\mathrm{d}^2 f(\hat{\eta})}{\mathrm{d}\hat{\eta}^2}\right|_{\hat{\eta}=0} \cdot \hat{\eta}^2 = \\
&= \sqrt{\frac{D^2}{4} + \eta_0^2} + \frac{\eta_0}{\sqrt{\frac{D^2}{4} + \eta_0^2}}\hat{\eta} + \frac{D^2}{8\sqrt{\frac{D^2}{4} + \eta_0^2}^3}\hat{\eta}^2 = \\
&= \frac{MD}{\sqrt{4M^2 - m^2}} + \frac{m}{2M}\hat{\eta} + \frac{D^2}{8}\frac{\sqrt{4M^2 - m^2}^3}{M^3 D^3}\hat{\eta}^2 = \\
&= \frac{D}{\sqrt{4 - \left(\frac{m}{M}\right)^2}} + \frac{m}{2M}\hat{\eta} + \frac{\sqrt{4 - \left(\frac{m}{M}\right)^2}^3}{8D}\hat{\eta}^2.
\end{aligned}$$

Consequently,

$$\begin{aligned}
U &\approx 2Mg\left(\frac{D}{\sqrt{4 - \left(\frac{m}{M}\right)^2}} + \frac{m}{2M}\hat{\eta} + \frac{\sqrt{4 - \left(\frac{m}{M}\right)^2}^3}{8D}\hat{\eta}^2 - \frac{L}{2}\right) - mg\left(\frac{mD}{2\sqrt{4M^2 - m^2}} + \hat{\eta}\right) = \\
&= \frac{gD}{\sqrt{4 - \left(\frac{m}{M}\right)^2}}\left(2M - \frac{m^2}{2M}\right) - MgL + \frac{Mg}{4D}\sqrt{4 - \left(\frac{m}{M}\right)^2}^3 \hat{\eta}^2 = \\
&= \frac{1}{2}MgD\sqrt{4 - \left(\frac{m}{M}\right)^2} - MgL + \frac{Mg}{4D}\sqrt{4 - \left(\frac{m}{M}\right)^2}^3 \hat{\eta}^2.
\end{aligned}$$

We see that an expression for the potential energy contains only absolute term, which shifts equipotential surfaces by a constant value, tus it does not affect movement, and a quadratic term, which corresponds to a harmonic oscillations. Having applied an analogy with a potential energy of a spring $E_{\text{pot}} = \frac{1}{2}kx^2$, we can talk about an effective „stiffness“ of the system

$$k = \frac{Mg}{2D}\sqrt{4 - \left(\frac{m}{M}\right)^2}^3.$$

Now, let’s investigate the kinetic energy. It has a form of $E_{\mathrm{kin}} = \frac{1}{2}\mu u^2$, thus we can refer to an efective mass of the system in the equilibrium state

$$\mu = \frac{2M\eta_0^2}{\frac{D^2}{4} + \eta_0^2} + m = 2M\left(\frac{m}{2M}\right)^2 + m = m\left(\frac{m}{2M} + 1\right).$$

An angular frequency of an oscillator with a stiffness k and mass μ is given by

$$\omega = \sqrt{\frac{k}{\mu}}.$$

After substituting of the expressions for k and μ and performing long disgusting arithmetics, we obtain

$$\omega = \sqrt[4]{(2M)^2 - m^2}\sqrt{\frac{2}{m} - \frac{1}{M}}\sqrt{\frac{g}{D}}.$$

Having made a use of $\frac{2M}{m} = \frac{41}{9}$, the results simplifies to

$$\omega = \frac{16}{3}\sqrt{\frac{10}{41}}\sqrt{\frac{g}{D}}.$$

Finally, the sought period is

$$T = \frac{2\pi}{\omega} = \frac{3\pi}{8}\sqrt{\frac{41}{10}}\sqrt{\frac{D}{g}}$$

and for $D = 25$ m,

$$T = \frac{3\sqrt{41}}{16}\pi \text{ s}.$$

Odpowiedzi

1 147 s

2 225

3 6

4 $\sqrt{2}m$

5 $4R$

6 6 cm

7 60 kg

8 10 cm

9 $\cot 42° \cdot \cos 45° \cdot 10\ \mathrm{m}^2 \doteq 7{,}85\ \mathrm{m}^2$

10 *Przedział* $\{54\ \mathrm{km/h}, 150\ \mathrm{km/h}\}$

11 $\dfrac{4\pi R^{3/2}}{3\sqrt{Gm}}$

12 $\sqrt{v^2 + 2gH}$

13 60°

14 $\arcsin\dfrac{1}{4} \doteq 14{,}48°$

15 50 h

16 $\dfrac{24\pi}{3\pi + 8} r\rho \doteq 4{,}327 r\rho$

17 13

18 Możliwe rozwiązania: *03:14:54, 08:45:06, 15:14:54, 20:45:06.*

19 $\left(1 - 4 \cdot 3^{-13/4}\right) \rho_v \doteq 887\ \mathrm{kg/m^3}$

20 $\sqrt{2\mu g L \left(\frac{m+M}{M}\right)}$

21 3

22 $90°/k, k \in \mathbb{N}$

23 *Akceptowalne są rozwiązania z przedziału* 2,42 – 2,48 s.

24 $\frac{42\pi^2 + 60\pi}{24 + 28\pi + 7\pi^2} r\rho \doteq 3{,}3306 r\rho$

25 25 mm

26 0 N, nie musi wywierać żadnej siły.

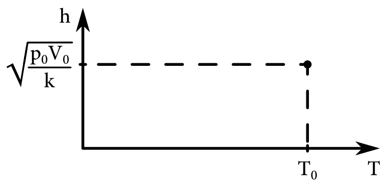
27 $\frac{3v^2}{4g} + r(1 - \cos\alpha)$

28 σa^4

29 16

30 $4\pi\left(2 - \sqrt{2}\right)$ sr, *or* $\left(2 - \sqrt{2}\right)$ of space.

31 *Rozwiązaniem jest pojedynczy punkt.*



32 $I_0\left(\cos\frac{\pi}{20}\right)^{20} = I_0\left(\cos 9°\right)^{20} \doteq 0{,}78\, I_0$

33 $\sqrt{\frac{2}{m}\left(\frac{5}{2}p_0V_0 + E\right)\left[1 - \left(\frac{V_0}{LS}\right)^{2/5}\right]}$

34 Niny, n^2-razy

35 2,44 m

36 278,785 K $\doteq$ 6,635 °C. *Akceptowalne są rozwiązania różniące się o mniej niż 0,5 K.*

37 11

38 $I_0 \cdot 0{,}9^5 \cdot \left(\cos\frac{\pi}{10}\right)^{10} = I_0 \cdot 0{,}9^5 \cdot \left(\cos 18°\right)^{10} \doteq 0{,}36\, I_0$

39 $1{,}31 \times 10^{-7}$ C

40 $\frac{3\sqrt{41}}{16}\pi$ s