Milí čitatelia,

v rukách držíte zbierku úloh 22. ročníka Fyzikálneho Náboja. V zbierke sa nachádzajú všetky úlohy, s ktorými ste sa v roku 2019 mohli na súťaži stretnúť. K úlohám prikladáme aj vzorové riešenia, z ktorých sa môžete mnohé naučiť. Ak by ste danému vzorovému riešeniu nerozumeli, neváhajte sa nám ozvať, všetko objasníme.

Fyzikálny Náboj po minulom roku pokračuje vo svojej medzinárodnej tradícii. V roku 2019 sa do Náboja zapojili okrem Bratislavy takisto mestá Košice, Praha, Ostrava, Budapešť a Gdańsk. Výsledky vzájomného súboja si môžete pozrieť na našich stránkach.

Táto zbierka by nikdy nevznikla bez výraznej pomoci mnohých ľudí, ktorí sa koniec koncov podieľali na celom vývoji Fyzikálneho Náboja. Všetci sme študentmi Fakulty matematiky, fyziky a informatiky Univerzity Komenského a väčšina z nás sa aj aktívne podieľa na organizovaní Fyzikálneho korešpondenčného seminára (FKS).

FKS je korešpondenčný typ fyzikálnej súťaže. Zhruba raz za mesiac zverejňujeme rôzne zaujímavé fyzikálne úlohy, ktorých riešenia nám posielate do určených termínov. My vám za to dávame adekvátne body a tých najlepších pozývame koncom zimného a letného polroka na týždňové zážitkové sústredenie. Viac informácií nájdete na stránke `https://fks.sk/`.

Za finančnú pomoc ďakujeme firmám ESET a PosAm a za medzinárodnú spoluprácu lokálnym organizátorom: Róbert Hajduk (za Košice), Daniel Dupkala (za Prahu), Lenka Plachtová (za Ostravu), Ágnes Kis-Tóth (za Budapešť) a Kamil Żmudziński (za Gdańsk). V mene celého organizačného tímu veríme, že ste si v roku 2019 Fyzikálny Náboj užili a dúfame, že vás všetkých uvidíme aj o rok! Či už v roli súťažiacich, alebo organizátorov (v prípade, že už budete vysokoškolákmi).

Jaroslav Valovčan

Hlavný organizátor

Zbierku zostavili:

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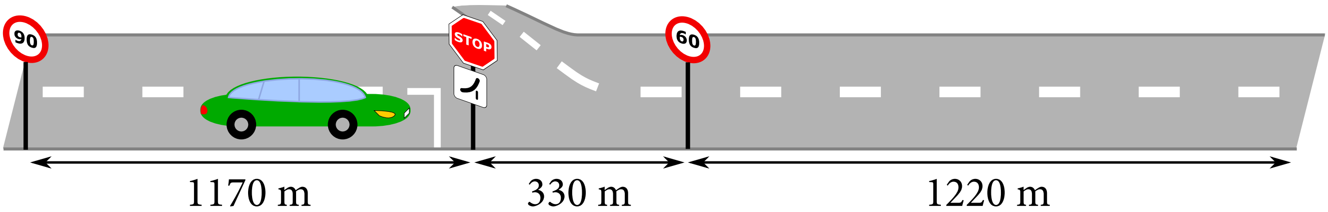
Katarína Dančejová
Justína 'Plyš' Nováková
Peter 'Bubu' Onduš
Katarína Častulíková
Martin 'Kvík' Baláž

Výsledky súťaže, archív úloh a ďalšie informácie nájdete na stránke `https://physics.naboj.org/`.

Problems

1 Martin has bought a new car. It can accelerate from 0 to 100 km/h in 10 seconds and decelerate from 100 km/h to 0 in 6 seconds. What is the shortest time Martin needs to travel the road in the picture, if he obeys all traffic regulations?

Of course, at the start and at the end of the road, his car must not move. Speed limits on the traffic signs are in km/h.



2 Kate found a nice section of highway running along a railway. Sometimes, she drives there to race against an express train. The express train has 150 times higher momentum, but only 100 times higher kinetic energy than Kate's car. What is the ratio of masses of the train and the car?

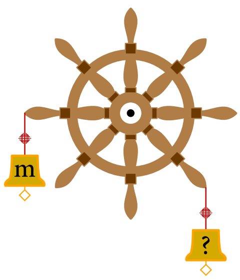
3 Matt was strolling at a fair, when he spotted a huge balloon filled with helium, which had radius of 50 cm and weighed 20 g when empty. He loved it and bought it right away… but then he started worrying it could float away.

Tying it to something heavy seemed too boring. So he decided to inflate some smaller balloons, which weighed 10 g when empty and had a radius of 20 cm, with air and tie them to the helium balloon. At least how many of these smaller balloons did he need to keep the helium balloon from floating away?

The elasticity of the balloons can be neglected.

4 Justine was on a cruise. She came back home with a decorative ship's wheel. She put a nail to the wall and hanged the center of the wheel on it. As she did not have enough wall space for new souvenirs, she hanged a Chinese bell with mass m on the leftmost spoke of the rudder.

Naturally, it rotated the wheel a bit, so she had to hang another bell on the right bottom spoke. What should the second bell's mass be so that that the wheel returns to its original position?



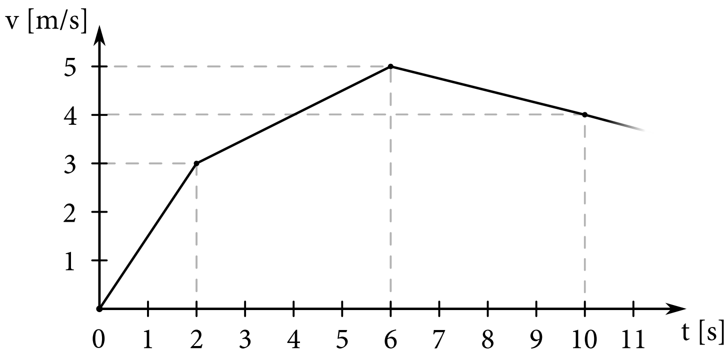
5 Two Scotsmen are fighting for a one-penny coin. Each of them is pulling one end, until they form a wire with radius r, length l and resistance R. However, it does not stop them from pulling. What will the resistance of the wire be once they double its length?

Assume that a volume of the coin does not change due to tension.

6 Teresa likes flags. She decided to sew her own flag of Seychelles. When she was done, she noticed that its centre of mass is not exactly in the geometric centre of the flag. By weighing the fabric she had used for each section of the flag, she found out that densities of fabrics from left to right were 600, 400, 300, 500 and 900 g/m^2. How far is the centre of mass of Teresa's flag from its geometric centre, if its size is 180 cm × 90 cm?

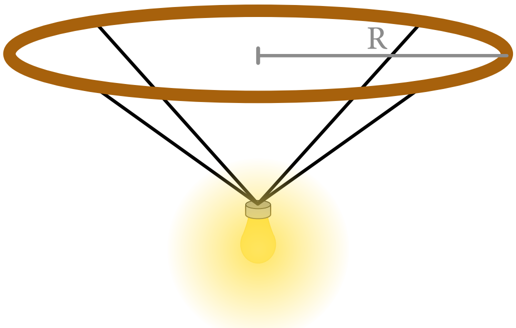
The seams divide the top and right side of the flag to thirds.

7 Isaac is pulling Teresa on her sled with constant force F. The sled with Teresa has mass $m = 60$ kg. At first, Isaac is pulling the sled across a frozen lake without any friction. Then he moves on the ground, where the friction between snow and the sled is $f > 0$. After a while, Peter jumps on the sled too. The velocity of the sled is shown in the graph. What is Peter's mass?



8 Martin has moved to a new empty apartment. As an astronomer he does not have much money, and since he also wants to eat at least once in a while, he decided to construct his own chandeliers.

His chandelier consists of a hoop with radius $R = 24$ cm that hangs from the ceiling. Then he attaches a lightbulb with mass $m = 1$ kg to the hoop using four evenly ditributed massless strings. Because of the low budget, the strings are not very strong and they can only withstand tensile force $F = 6.5$ N. What is the smallest height at which the lightbulb can be hanging below the hoop, if the strings must not snap?



9 Andrew erected a high flagpole in Bratislava. On its top there is a monumental flying flag with area of 10 m^2. It is noon of the autumnal equinox and the wind is blowing from the northwest. What is the area of the shadow the flag casts on the ground?

Bratislava is located on 48th parallel north.

10 Jacob was travelling on a train to visit his grandparents. As he got bored, he started counting the cars the train was passing on the highway next to the railway. He saw four times more cars per unit of time in the opposite direction than in the direction of his travel. What were the possible speeds at which Jacob's train was travelling?

Traffic on the highway was equal in both directions and the cars' speed was 90 km/h.

11 Two moons of mass m share the same orbit around a planet of mass $2m$. All three objects lie on a straight line at any time.

Kate measured that the radius of the moons' orbit is R. How long does it take them to orbit the planet?

12 On warm summer nights stray cats keep making noise under the dorms. On one particularly hot and sleepless night Thomas lost his temper, got out on the balcony at height H above the ground and started throwing rotten eggs at the cats. All eggs always have the same initial velocity v, but are thrown under various angles α.

What is the lowest velocity the egg can have at the moment when it hits the ground, if air resistance is neglected?

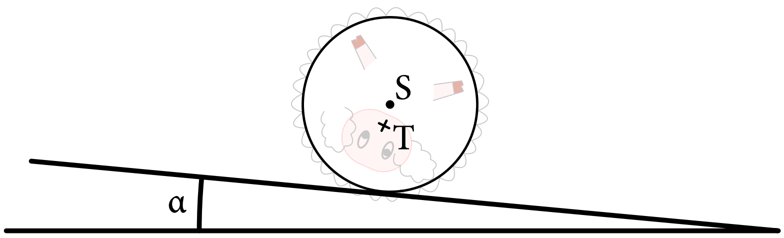
13 Andrew and Danny recently got their skipper licences. They rented boats and went sailing over the seven seas. However, their skills are still rather poor, so for safety reasons they only sailed forward at constant velocity.

- When Andrew crossed Danny's path, their mutual distance was $3d$.
- After time t, Danny crossed Andrew's path and their mutual distance was $4d$.
- $5t$ later, they both ran out of fuel and they immediately stopped at mutual distance $21d$.

At what angle do their paths intersect each other?

14 Ever since George received his master's degree in physics, his shepherding endeavours look very strange to outside observers. He approximates his sheep as homogeneous cylinders with radius R and mass m. When a sheep finishes grazing on the hill, it pulls its legs in and rolls down.

It is not easy, though. Once the sheep has eaten enough, its centre of mass T moves by $R/4$ from its centre S in radial direction. What is the lowest inclination of the hill so that the sheep certainly roll down, irrespective of their initial angle of turn?



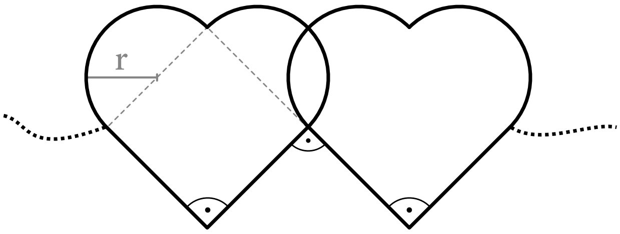
15 Little princess Michaela is sitting on her little planet P-314 and is very upset. The planet's star only sets every 60 hours, which means that to see enough romantic sunsets, she has to keep shifting her heavy wooden chair. On the other hand, a year is only 300 hours long, hence at least she gets to celebrate many birthdays.

After one birthday party she pulled her chair to the night side of the planet and looked up to the stars. In a while she realised that those distant stars seem to orbit the planet with a different period than her sun. What is the actual period of rotation of the planet?

Assume that the planet orbits the star in the same direction as it rotates and that its orbit is circular. The axis of rotation is perpendicular to the orbital plane.

16 Francis bought Theresa for her birthday a necklace composed of two same symmetric hearts. Theresa was obviously very happy, but since she is a physicist, she was amazed not only by the aesthetics but also by electrical properties of the jewel.

She found out that the resistance per unit length of the wire, from which the necklace is made, is ρ. What is the total resistance of the jewel between points where the jewel is connected to the chain? The radius of curvature of circular parts of the necklace is r.



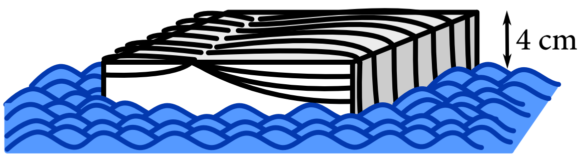
17 Luke enjoys building simple domestic appliances. Last time, he found three resistors with resistances 20, 30 and 60 Ω and assembled an electric heater from them. However, he is not very good in product design. When he wants to change the power of the heater, he has to take it apart it and re-connect the resistors. How many different power settings (not including the off state) can he achieve if the main circuit-breaker in his house is rated at 15 A?

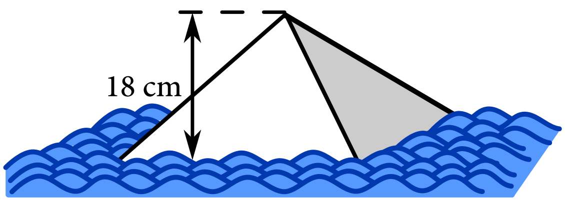
Mains voltage is 230 V. Luke does not have to use all resistors each time.

18 Christian has a beautiful wall clock. The clock has only an hour-hand and a minute-hand, which move by small steps each second. The hour-hand has length d and mass $2m$, the minute-hand has length $2d$ and mass m. In which second of the day do the hands exert maximum torque in the centre of the clock?

Find at least one solution. Submit it in form `HH:MM:SS`. *Do not be afraid to use a calculator.*

19 Derek the cube likes floating in the pool. If he lies so that his base is parallel to the water surface, he protrudes 4 cm above the surface. If he turns, so that one of its body diagonals is perpendicular to the water surface, the visible part of him is a three-sided pyramid sticking 18 cm above the surface. What is Derek's density?





20 Matthew took a rock of mass m and put it on the front end of a skateboard of length L and mass M. There is no friction force between skateboard and the ground while the coefficient of friction force between the rock and the skateboard is μ. Matthew shoved the skateboard, so that it started moving with velocity v in the longitudal direction. What is the highest possible velocity v, so that the rock remains on the skateboard?

Assume that the skateboard was accelerated instantaneously, whilst the rock remained at rest with respect to Matthew at the first moment.

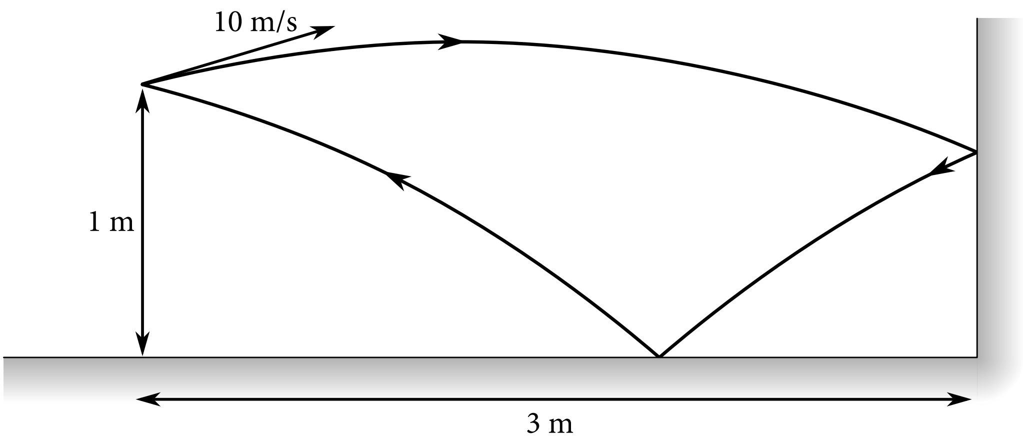
21 Thomas accidentally nudged his huge smartphone with height of $l = 16$ cm off the table. Fortunately, he managed to give it a spin before it was too late. As he owned a modern phone with the display covering the entire front side, his only hope was that the phone would land directly on its back side, otherwise the display would likely crack.

The height of the table was $h = 0.8$ m and the phone was originally lying on its back side. At most how many rotations could the phone perform in the air if it landed on its back side?

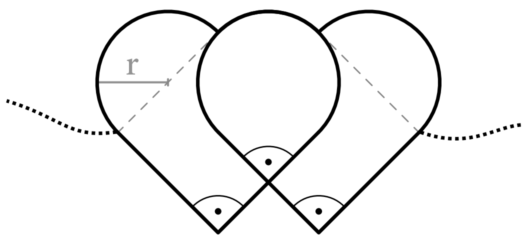
Assume that no impulse in a vertical direction was given to the smartphone when giving it a spin.

22 Irene took two mirrors and placed them on the table so that the angle between their reflecting surfaces was α. Then, she pointed a laser pen inside so that the beam was parallel to the one of the mirrors. She noticed that the beam returned along the same trajectory as it came in. What are the possible values of angle α?

23 Charlie unexpectedly found a bouncy ball in his pocket. Without thinking, he threw it from height $h = 1$ m to a wall in distance $d = 3$ m with velocity $v = 10$ m/s. The bouncy ball hit the wall, then bounced off the floor and jumped right back to Charlie's hand. How long did the entire journey of the bouncy ball take if all its bounces were perfectly elastic?



24 Theresa was so delighted with the gift from Francis, that now she loves him even more. For obvious reasons this required a few changes in the necklace. It still consists of two same symmetric hearts. The resistance per unit length of the material is still ρ and the radius of circular parts is r. Find out what is the resistance of the necklace between the ends now.

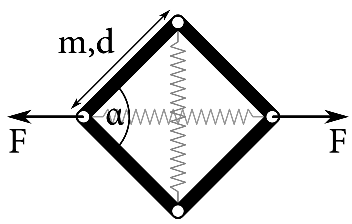


25 Matthew has a big metal barrel, half-full of warm tea. The barrel has a faucet at the bottom, so the tea can be tapped. The height of the kettle is 1 m. Matthew started tapping the tea out, however it stopped pouring after a while, as no air could get inside. How much could the tea's surface go down without letting any air into the kettle? The air inside the kettle had standard atmospheric pressure before the tap was opened.

Give the result with accuracy of 1 mm.

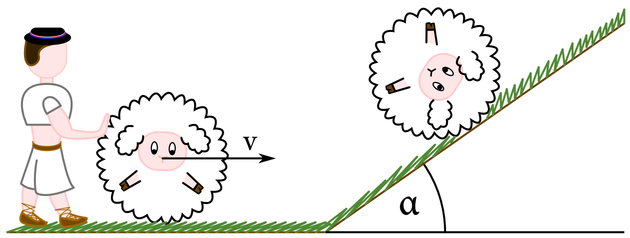
26 Michelle loves creating mechanical systems using springs. She bought a square frame with each side of mass m and length d. The sides of the frame are connected with knuckles in the corners, so when bending, the whole frame is in the same plane.

Michelle put the frame horizontally on the ground. Then she took two springs with the unstretched length zero and spring constant k and stretched them on the diagonals of the frame. Then she started pulling two opposite corners away from each other. What force F she has to pull the corners away with, so the internal angle nearest to her hand is α?

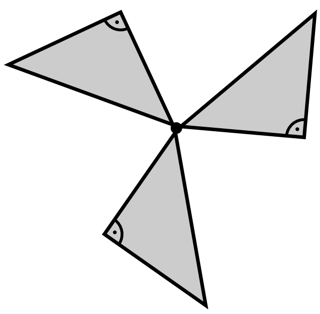


27 When George found out the sheep roll down the hill safely, he got another idea of how to use their cylindrical shape for the good. He starts rolling the sheep on the horizontal plane with velocity v. The plane smoothly changes to a hill with inclination α.

To which height above the plane will the grass on the hill be rolled over when the sheep stops rolling? The radius of a hungry sheep is r.

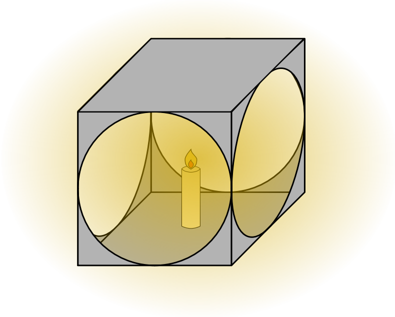


28 Stella investigates alternative energy sources. She bought a lot of metal sheets with area density σ in bargain sale and welded a simple propeller from them. The propeller has three blades, each in the shape of an isosceles right triangle with leg length a. The blades are attached to the hub uniformly every 120°. What is the moment of inertia of Stella's propeller with respect to the axis of the hub?



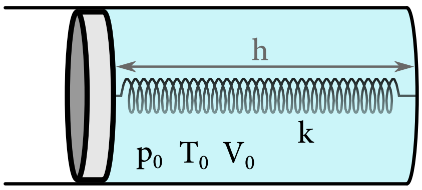
29 Horatio found a 1 m long spring. Applying certain secret scientific methods, he found out that the stiffness of the spring is 70 N/m. He wants to cut the spring into several smaller pieces of an equal length, so that he can put a wooden platform with mass 10 kg on them. To how many little pieces must he cut the spring, so that they can keep the platform with Horacio above the ground? Horatio's mass is 100 kg.

30 When tidying up the room, a magic lantern was found. It has a cubic shape and a light is emitted only through four apertures in side faces of the lantern, which have shape of an inscribed circles. What solid angle can be illuminated by the lantern if the source of light is exactly in the centre of the lantern.



31 Oliver took a long cylindrical container and closed it with a piston. He attached the piston to the bottom of the container using a spring, which has stiffness k and zero rest length. The piston enclosed ideal gas with volume V_0, pressure p_0 and temperature T_0 in the container.

At that moment, he realised that the length of the spring h is a great state quantity. He immediately drew an h–T diagram of an adiabatic process with an ideal gas inside the piston. Assume that no external force is allowed to act on the piston, so the only forces acting on the piston are the pressure force of the gas inside and the force from the spring. Draw the corresponding diagram and mark all important values.



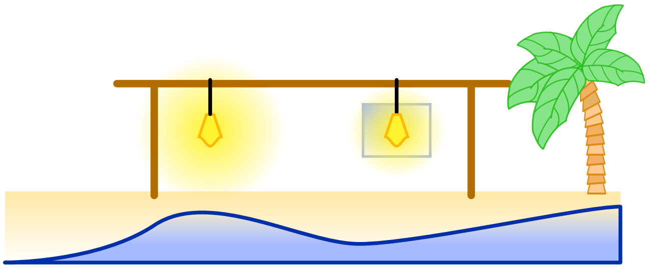
32 Simon urgently needs horizontally polarised light. However, he only has a source of vertically polarised light with intensity I_0 and 10 perfect polarising filters. He wants to arrange the filters in such a way that they turn a plane of polarisation. What is the highest intensity of the horizontally polarised light behind the filters, if the light cannot contain any vertically polarised component?

Perfect polarising filter lets all light in direction of polarisation through and blocks all light in direction perpendicular to it.

33 Adam decided to destroy one of the institutions in his neighborhood. The best way how to do it is to use a cannon. He took the cannon with a barrel of length L with a cross-section S. He inserted a projectile with mass m, so that there was still some air with volume V_0 and pressure p_0 left behind it. An explosion of gunpowder supplied the air behind the projectile with energy E. The air started to expand adiabatically, which resulted in firing of the projectile out. What was the velocity of the projectile when leaving the barrel? Assume the air to be an ideal two-atom gas.

34 Sirens Hannah and Nina are trying to lure some poor sailors using the light of their two identical lanterns. Nina hung the lantern on the holder, from where it shines isotropically with power P. However Hannah put her lantern inside a huge cube made of glass with refractive index n, and positioned it so that one of its sides faces the sea.

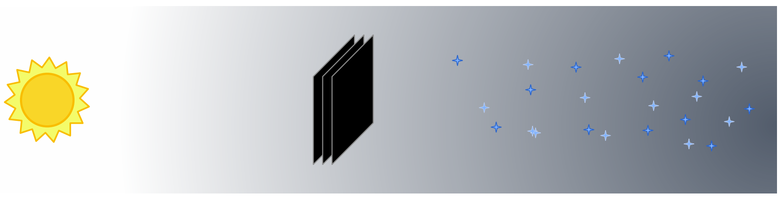
Which lantern will the sailors see as brighter now and by what factor, if they are looking perpendicularly to the shore from a large distance?



35 Justine likes playing with particles. She installed her own cloud chamber and placed it inside strong magnetic field with intensity $B = 10$ T. She noticed that at a certain moment, a proton and a muon entered the chamber, having recorded their paths of shape of two concentric half-circles in fog. Both particles were moving with velocity $v = 0.99c$ perpendicularly to the direction of the field. Their paths were parallel before entering the chamber. What was their mutual distance at that time?

Muon has same charge as electron, but it is 207-times heavier.

36 There are three big, perfectly black blankets drying on ropes behind the window of a space station. The Sun is shining on the first blanket perpendicularly with power per unit area of $F_{\odot} = 1370\ \mathrm{W/m^2}$. What is the equilibrium temperature of the third blanket if there is only empty outer space behind it?



37 Matthew decided to walk the famous Little Carpathian Wine Route. But already in the first wine cellar he was more amazed by the wine barrels than by their content. They were cylindrical in shape, with radius of the base 0.5 m and height 1.6 m. The barrels consisted of straight battens, which were held together by metal rims. He wondered, how many metal rims at least are needed, so the barrel does not fall apart, when it stands on its base.

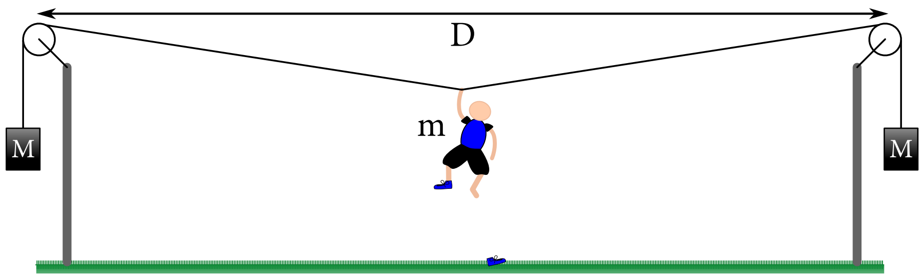
Ultimate tensile strength of the metal rim is 20 MPa, *its cross-section is* 30 mm^2 *and density of the wine is* 1000 kg/m^3.

38 Simon is still playing with polarising filters. Naturally, after so many experiments, they got a bit dirty. Now, each of them lets pass only 90 % of incoming light in the plane of polarisation (but still no light in the perpendicular plane).

What is the highest intensity of horizontally polarised light which Simon can get at the output of his apparatus? Entering light is still vertically polarised and its intensity is I_0. Simon still has 10 polarising filters, but he need not use all of them.

39 Mary was inflating soap bubbles. She is a good-willed and sympathetic person, anmd she did not like that the air inside was oppressed. Because of that she decided to bring electric charge onto a bubble. What charge did she have to bring on the bubble with radius 4 cm, so that there is an atmospheric pressure inside? Assume that a surface tension of soap water is equal to 1/3 of surface tension of pure water. Submit the result rounded to three significant digits.

40 Jacob constantly improves his beloved railroad scale model. Last time he experimented with overhead lines. The load-bearing cable is hung through a pair of pulleys placed at the top of the poles, and it is tensioned with heavy weights attached to its ends. During a routine inspection of his lines, Jacob suffered a mysterious accident, which left him swinging on the line, right in the middle between two poles. Find the period of his vertical motion. Assume that $\frac{M}{m} = \frac{41}{18}$ and that the poles are spaced $D = 25$ m apart.



Solutions

1 First of all we express the acceleration and deceleration of the car in units that are much easier to use:

$$a_+ = \frac{100\ \mathrm{km/h}}{10\ \mathrm{s}} = 10\frac{\mathrm{km/h}}{\mathrm{s}} \qquad \text{a} \qquad a_- = \frac{100\ \mathrm{km/h}}{6\ \mathrm{s}} = \frac{100}{6}\frac{\mathrm{km/h}}{\mathrm{s}}.$$

To avoid manipulating ugly numbers, we keep accelerations in this form and we transform time so that it fits these units. Hence, when given acceleration and time, the expression at^2 takes units

$$\frac{\mathrm{km/h}}{\mathrm{s}} \cdot \mathrm{h} \cdot \mathrm{s} \equiv \frac{\mathrm{km/h}}{\mathrm{s}} \cdot 3600\ \mathrm{s} \cdot \mathrm{s}.$$

For clarity, let us divide the path to segments between traffic signs:

I. First we need to accelerate from rest to 90 km/h, which takes 9 s. The distance covered within this period is

$$s_{\mathrm{Ia}} = \frac{1}{2}a_+t^2 = 112.5\ \mathrm{m}.$$

We have to stop at the end of the first segment. Deceleration takes 5.4 s and the distance is

$$s_{\mathrm{Ic}} = v_{\mathrm{I}}t - \frac{1}{2}a_-t^2 = 60\ \mathrm{km/h} \cdot \frac{5.4\ \mathrm{s}}{3600\ \mathrm{s/h}} - \frac{1}{2} \cdot \left(\frac{100}{6}\frac{\mathrm{km/h}}{\mathrm{s}}\right) \cdot \frac{(5.4\ \mathrm{s})^2}{3600\ \mathrm{s/h}} = 67.5\ \mathrm{m}.$$

The remaining part of the first segment $s_{\mathrm{Ib}} = 990$ m is travelled at constant speed. Therefore the resulting time to travel the first segment is

$$t_{\mathrm{I}} = 9\ \mathrm{s} + \frac{s_{\mathrm{Ib}}}{v_{\mathrm{I}}} + 5.4\ \mathrm{s} = 54\ \mathrm{s}.$$

II. The beginning is same as in the first segment. Namely, we accelerate in 9 s and we cover a distance of $s_{\mathrm{IIa}} = 112.5$ m. At the end, we have to slow down to 60 km/h. It takes time 1.8 s and the car travels a distance of

$$s_{\mathrm{IIc}} = 90\ \mathrm{km/h} \cdot 1.8\ \mathrm{s} - \frac{1}{2} \cdot \frac{100}{6}\frac{\mathrm{km/h}}{\mathrm{s}} \cdot (1.8\ \mathrm{s})^2 = 37.5\ \mathrm{m}.$$

The rest of the segment is travelled at constant speed, thus resulting time is

$$t_{\mathrm{II}} = 9\ \mathrm{s} + \frac{s_{\mathrm{IIb}}}{v_{\mathrm{I}}} + 1.8\ \mathrm{s} = 18\ \mathrm{s}.$$

III. Deceleration at the end of the ride takes 3.6 s and corresponding distance is

$$s_{\mathrm{IIIb}} = v_{\mathrm{III}}t - \frac{1}{2}a_-t^2 = 30\ \mathrm{m}.$$

The remaining part of the segment is $s_{\mathrm{IIIa}} = 1190$ m long. We pass it with constant speed, thus in the third segment we spend time

$$t_{\mathrm{III}} = \frac{s_{\mathrm{IIIa}}}{v_{\mathrm{III}}} + 3.6\ \mathrm{s} = 75\ \mathrm{s}$$

Summing it all together, the total time of the drive is 147 s.

2 Let's mark mass of the train M, speed of the train V and mass and speed of Kate's car m and v. From the task we know that the ratio of their momenta is

$$\frac{MV}{mv} = 150, \tag{2.1}$$

from which we can figure out the ratio of the speeds

$$\frac{V}{v} = 150\frac{m}{M}. \tag{2.2}$$

The ratio of the kinetic energies is

$$\frac{\frac{1}{2}MV^2}{\frac{1}{2}mv^2} = 100. \tag{2.3}$$

When we use the ratio of speeds from the equation 2.2, put it into the following equation 2.3, we get a result

$$\frac{M}{m} = 100\frac{v^2}{V^2} = \frac{100}{150^2}\frac{M^2}{m^2}, \qquad \text{from where} \qquad \frac{M}{m} = 225.$$

3 Let us denote the mass of the helium balloon M, smaller balloons m and their radii R and r respectively. The balloon cluster does not float away, if its total weight is not smaller than the lifting force acting on it. When we realise that weight acting on the air inside the smaller balloons balances the lifting force acting on smaller balloons, we get a condition for not floating away

$$M + Nm + \frac{4}{3}\pi R^3 \rho_{\mathrm{He}} \geq \frac{4}{3}\pi R^3 \rho_a.$$

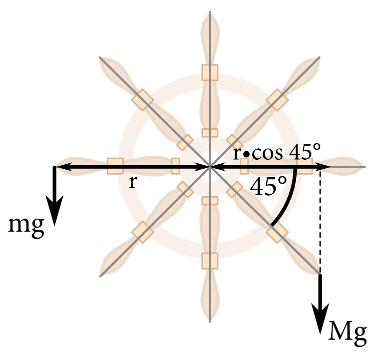
For the lowest number of balloons we get

$$N = \left\lceil \frac{\frac{4}{3}\pi R^3 \left(\rho_a - \rho_{\mathrm{He}}\right) - M}{m} \right\rceil.$$

When we put in the numerical values, we obtain $N = 6$.

4 To prevent the ship's wheel from rotating, the total torque calculated with respect to the wheel's axis has to be zero. Therefore

$$mgr = Mgr\cos 45° \quad \Rightarrow \quad M = m\frac{1}{\frac{\sqrt{2}}{2}} = \sqrt{2}m.$$



5 The resistance of the wire depends on its length, cross-section and material's resistivity ρ as

$$R = \rho\frac{l}{S}.$$

Plastic deformation of the wire does not change its volume nor specific resistivity. If the length increases to twice its original value, the cross-section will decrease to one half because of conservation of volume. Its resistance is thus

$$R' = \rho\frac{2l}{\frac{S}{2}} = 4\rho\frac{l}{S} = 4R.$$

6 Denote area densities of individual triangles gradually from left σ_1 to σ_6. Their masses are calculated simply by multiplying the densities by their areas. We obtain

$$m_1 = 162\ \text{g}, \quad m_2 = 108\ \text{g}, \quad m_3 = 81\ \text{g}, \quad m_4 = 81\ \text{g}, \quad m_5 = 135\ \text{g}, \quad m_6 = 243\ \text{g}.$$

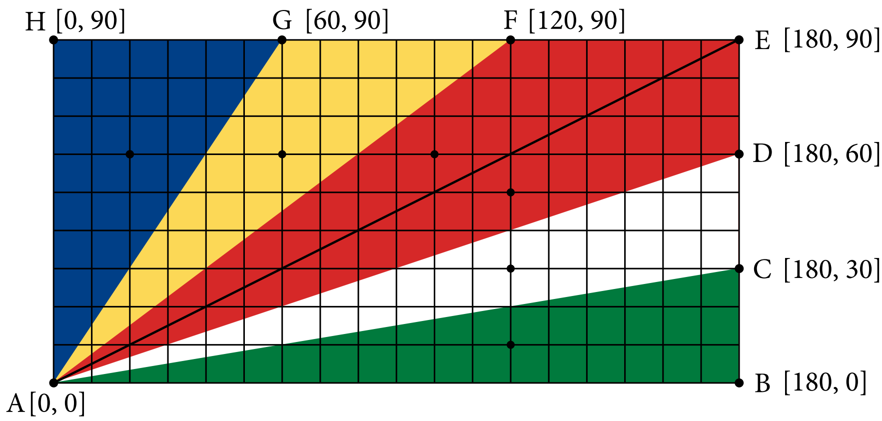


Figure 6.1: *Vlajka na štvorcovej sieti.*

Next, we find centres of mass of individual triangles. Let's begin with triangle $\triangle AGH$. The centroid, and thus also a centre of mass, of a triangle is at the intersection of its medians, which is in one third of any median measured from its base. If we place the triangle onto a regular grid, draw a median connecting vertex A with the midpoint of edge GH and find the point in two thirds of its length, we see that coordinates of the centroid are $[20\ \text{cm}; 60\ \text{cm}]$. Analogously we can find the coordinates of centroids of other triangles:

$$T_1 = [20\ \text{cm}; 60\ \text{cm}], \quad T_2 = [60\ \text{cm}; 60\ \text{cm}], \quad T_3 = [100\ \text{cm}; 60\ \text{cm}],$$

$$T_4 = [120\ \text{cm}; 50\ \text{cm}], \quad T_5 = [120\ \text{cm}; 30\ \text{cm}], \quad T_6 = [120\ \text{cm}; 10\ \text{cm}].$$

Consequently, we need to find the centre of mass of the whole flag. In general, the formula for the x-coordinate of the centre of mass of objects with masses m_1, m_2,…,m_n and with x-coordinates of their centroids x_1, x_2, …, x_n is

$$X = \frac{m_1x_1 + m_2x_2 + m_3x_3 + \ldots + m_nx_n}{m_1 + m_2 + m_3 + \ldots + m_n}.$$

In our case, we obtain $X = 90$ cm.

Analogously, the y-coordinate of the centroid can be calculated:

$$Y = \frac{m_1 y_1 + m_2 y_2 + m_3 y_3 + \ldots + m_n y_n}{m_1 + m_2 + m_3 + \ldots + m_n} \doteq 39\ \text{cm}.$$

The distance from the origin of the coordinate system is calculated applying the Pythagorean theorem as

$$l = \sqrt{X^2 + Y^2} \approx 98\ \text{cm}.$$

7 Accelerations of the sled in the individual segments can be determined from the graph:

$$a_1 = 1.5\ \text{m/s}^2, \qquad a_2 = 0.5\ \text{m/s}^2, \qquad a_3 = -0.25\ \text{m/s}^2.$$

In the first segment, Isaac pulls with force F. According to the second Newton's law

$$m = \frac{F}{a_1}. \tag{7.1}$$

In the second segment, a friction force $F_{t1} = fmg$ acts as well, where f is a friction coefficient between the sled and ground. The sum of forces acting on the sled is thus

$$F - F_{t1} = F - fmg = ma_2. \tag{7.2}$$

In the last segment, there are two differences – higher total mass $m + M$ and a higher friction force $F_{t2} = f(M + m)\,g$. We obtain

$$F - F_{t2} = F - f(M + m)\,g = (M + m)\,a_3. \tag{7.3}$$

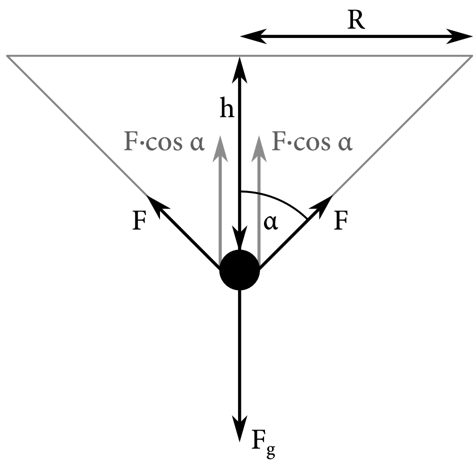
Now, we are able to determine everything we are interested in. From equation 7.1, we substitute for m in 7.2 and we obtain

$$f = \frac{a_1 - a_2}{g}.$$

Then we substitute for f in 7.3 and we finally obtain Peter's mass as

$$M = m\frac{a_3 - a_2}{a_2 - a_1 - a_3} = 60\ \text{kg}.$$

8 Since the chandelier is radially symmetric, we can consider just a simpler two-dimensional problem:



It is clear that the vertical component of the tension force in each string is $F\cos\alpha$. In order to prevent the chandelier from falling down, the resulting force acting on the bulb must be zero. As there are four strings attached to the bulb, we obtain

$$4F\cos\alpha = mg \quad \Rightarrow \quad \alpha = \arccos\frac{mg}{4F}.$$

We want to know height h. It can be expressed from the equation above, as

$$\tan\alpha = \frac{R}{h} \quad \Rightarrow \quad h = \frac{mgR}{\sqrt{16F^2 - m^2g^2}}.$$

After plugging in the numeric values, we obtain $h = 10$ cm.

9 Firstly, we calculate the height of the Sun on the sky in Bratislava. At noon, the Sun shines directly from south. On the day of equinox, the Sun is directly above the equator. The closest point of the Earth to the Sun is on the intersection of the equator and the meridian passing through Bratislava. Light beams arrive in Bratislava with angle of $90° - 48° = 42°$ with respect to the local surface.

As the wind blows from north-west, the flag is oriented in south-eastern direction, thus it is rotated by 45° with respect to sunbeams. Therefore, the cross-sectional area of the flag is effectively smaller when looking from the Sun. If we rotated the flag to the east-west direction, we would need a flag with a crossectional area only $\cos 45° \cdot 10\ \mathrm{m}^2$ to shade the same amount of sunbeams, as is shown on the left half of picture 9.1.

The sunbeams are projected on the ground. The length of the shadow in the east-west direction remains unchanged, whilst it is elongated in the north-south direction by factor h'/h. We get

$$h = h' \tan 42° \quad \Rightarrow \quad h' = h\cot 42°.$$

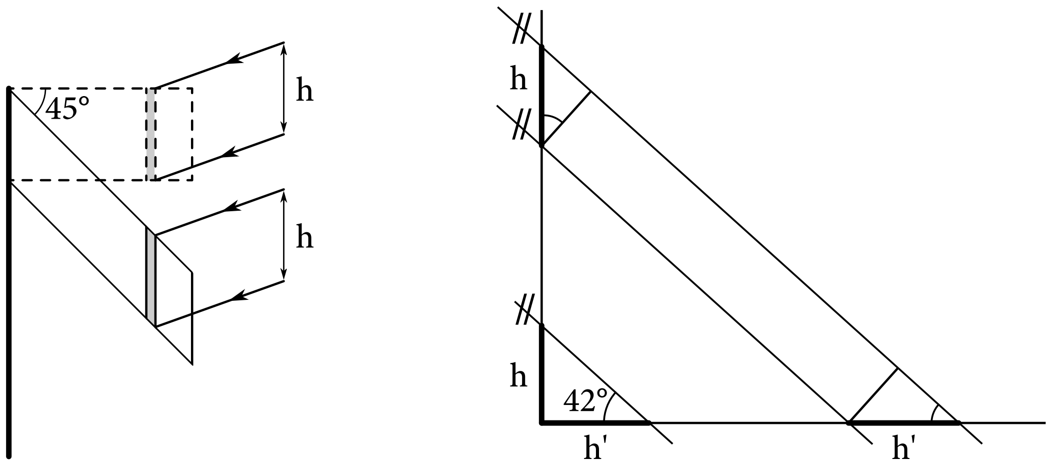


Figure 9.1: *The projection of the flag to east-west direction and to the ground.*

Therefore, the shadow on the ground has an area of $\frac{\cos 42°}{\sin 42°} \cdot \cos 45° \cdot 10\ \mathrm{m}^2 = \cot 42° \cdot \cos 45° \cdot 10\ \mathrm{m}^2 \doteq 7.85\ \mathrm{m}^2$.

Note that this is also valid for any non-rectangular shape of the flag – any shape can be arbitrarily well approximated by a large number of non-overlapping rectangles.

10 Denote an average distance between two consucutive cars d. Then an average time between meeting two consecutive cars is $t = \frac{d}{w}$, where w is a mutual velocity of cars and a train. A frequency is only a inverse value of the average time.

Let v be a velocity of cars and u a velocity of the train. Then a frequency of meeting cars driving in the opposite direction is $f^+ = \frac{v+u}{d}$ and cars driving in the same direction is $f^- = \frac{|v-u|}{d}$.[1] Obviously, $f^+ > f^-$. Suppose that f^+ is k-times greater than f^-. Then

$$v + u = k\,|v - u|\,.$$

We need to find a velocity of the train u satisfying the equation above. We have to consider two cases. If $v > u$, then $v + u = k\,(v - u)$, therefore

$$u = \frac{k-1}{k+1}v.$$

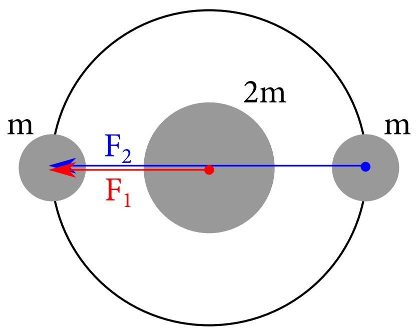
If $v < u$, then $v + u = k\,(u - v)$, thus

$$u = \frac{k+1}{k-1}v.$$

For $k = 4$ and $v = 90$ km/h, we obtain two possible velocities of the train $u = 54$ km/h and $u = 150$ km/h.

11 Each of the moons is attracted by gravitational force of the other moon and of the planet. In order to orbit the circular trajectory, the resulting force acting on the moon must be a cenripetal force corresponding to its radius R, e. i.

$$F_1 + F_2 = G\frac{Mm}{R^2} + G\frac{m^2}{4R^2} \stackrel{!}{=} \frac{mv^2}{R}.$$



In order to caculate the period, we need to express the speed of the moons. Consequently, the period is obtained as a ratio of covered distance and moon's speed

$$T = \frac{2\pi R}{v}.$$

Therefore

$$T = \frac{2\pi R}{\sqrt{\frac{9Gm}{4R}}} = \frac{4\pi R^{\frac{3}{2}}}{3\sqrt{Gm}}.$$

[1] An absolute value is present to secure that the frequency is always positive, as the train can be either faster or slower than cars.

12 During the throw, only the gravity acts on the egg, which is conservative – hence it can be described by a potential. Therefore, the problem can be solved in terms of kinetic and potential energy.

Thomas always throws the egg with same initial speed, regardless the direction. Therefore, a kinetic energy of the egg at tho moment of throwing is always

$$E_k(H) = \frac{1}{2}mv^2$$

and a potential energy is

$$E_p(H) = mgH$$

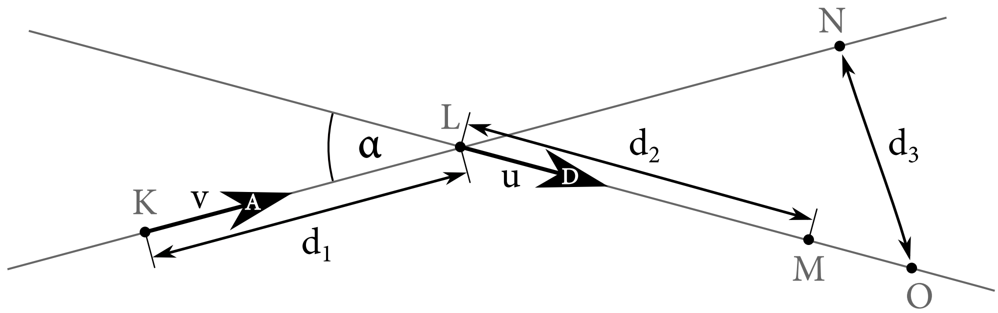
(considering a ground to be a level of zero potential energy).

A sum of the kinetic and potential energy is conserved, thus at the zero hight above the ground we get

$$E_k(0) + E_p(0) = \frac{1}{2}mv^2(0) + mg \cdot 0 \stackrel{!}{=} E_k(H) + E_p(H) = \frac{1}{2}mv^2 + mgH.$$

Therefrom, $v(0) = \sqrt{v^2 + 2gH}$, regardless the angle of elevation.

13 Let's draw the paths of boats:



Denote $d_1 = 3d$, $d_2 = 4d$ and $d_3 = 21d$. We can clearly see that

- a boat A from the point K to the point L gets in time t and covers distance d_1. We can easily calculate its velocity $v = \frac{d_1}{t} = \frac{3d}{t}$.
- a boat D from the point L to the point M gets in time t too and covers distance $d_2 = 4d$. Its velocity is $u = \frac{d_2}{t} = \frac{4d}{t}$.
- a sailing of the boat A from the point L to the point N takes time $5t$ and a covered distance is $s_1 = v \cdot 5t = 5d_1 = 15d$.
- a sailing of the boat D from the point L to the point O takes time $6t$ and a covered distance is $s_2 = u \cdot 6t = 6d_2 = 24d$.

We are to find an angle α. We have found the lengths of all sides of a triangle $\triangle LON$, thus the sought angle can be determined using the low of cosines

$$d_3^2 = s_1^2 + s_2^2 - 2s_1s_2\cos\alpha,$$

from where

$$\alpha = \arccos\frac{s_1^2 + s_2^2 - d_3^2}{2s_1s_2} = \arccos\frac{25d_1^2 + 36d_2^2 - d_3^2}{60d_1d_2}.$$

After substituting for d_1, d_2 and d_3 we obtain

$$\alpha = \arccos \frac{225d^2 + 576d^2 - 441d^2}{720d^2} = \arccos \frac{1}{2} = 60°.$$

$\boxed{14}$ If the sheep managed to climb the hill, a friction had to be sufficiently large, so that the sheep did not slip down. Therefore, in order the sheep to move, it has to be able to rotate around a point of contact with the hill. In order the sheep not to move, a torque calculated with respect to the point of cantact with the hill has to be non-negative, i. e. directed to the hill. The total torque is a result of three forces acting on the sheep:

- gravity acting in the centre of mass;
- normal force acting in the point of contact with the hill;
- frictional force acting also in the point of contact.

Forces acting in the point of contact do not contribute to the total torque, as a length of their moment arm is zero. The only force that has to be included into a calculation is gravity. In order the sheep to remain on the hill, the torque has to be zero. As the gravity always acts downwards, the centre of mass of the sheep has to be situated directly above the point of contact.

Maximal horizontal distance of a centre of the sheep and the point of contact, and hence the steepest slope corresponds to the case, in which the centre of mass of the sheep is in the same height as its geometric centre. An angle between the centre of mass, the point of contact and the geometric centre is equal to the slope of the hill α in this limit situation. Therefore

$$\sin \alpha = \frac{R/4}{R} = \frac{1}{4} \quad \Rightarrow \quad \alpha \leq \arcsin \frac{1}{4} \doteq 14.48°.$$

$\boxed{15}$ Denote individual times as follows:

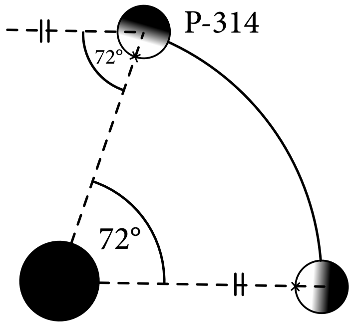
- time between two sunsets t_1;
- length of a year t_2;
- sought rotational period t.

The planet performs $\frac{t_1}{t}$ rotations and travels $\frac{t_1}{t_2}$ of its orbit between two consecutive sunsets. It corresponds to a state in which the planet has got a same orientation with respect to its sun again. A picture illustrate that the planet has performed one whole rotation and an additional fraction of another, which is exactly equal to the travelled fraction of orbit. In terms of math

$$\frac{t_1}{t} = 1 + \frac{t1}{t2}.$$

Therefrom

$$t = \frac{t_1 t_2}{t_2 + t_1} = 50 \text{ h}.$$



16 Let's mark the four important points on the necklace A, B, C a D.

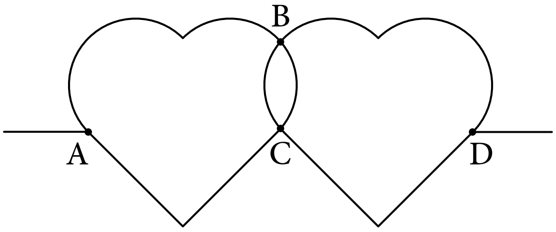


Figure 16.1: *Four important points on the necklace*

Effective resistance between points A and B is $\frac{3}{2}\pi r\rho$, and because of the symmetry, the resistance between B and D is the same. Similarly, the resistance between the points A and C as well as between C and D is $4r\rho$.

From the symmetry, it can be also concluded that the potential in the points B and C is equal and we can omit the wire between them. Therefore we get two parallel wires, one with the resistance of $8r\rho$ and the other with the resistance of $3\pi r\rho$.

The resistance of the whole circuit is therefore

$$\frac{1}{\frac{1}{8r\rho}+\frac{1}{3\pi r\rho}} = \frac{24\pi}{3\pi+8} r\rho \doteq 4.327r\rho.$$

17 Since voltage in the circuit is constant, the power output of the heater only depends on its resistance. Power can be expressed as the product of voltage and current; however, current flowing through the circuit is uniquely determined by the value of resistance as $I = U/R$. Hence

$$P = UI = \frac{U^2}{R}.$$

If we enumerate all possible ways how Luke can connect the resistors, we will find out there are 18 distinct schemes:

- no resistors (one scheme),
- a single resistor (three ways how to choose it),
- two resistors in series (three ways which to omit, order is not important),
- three resistors in series (only one way, order is not important),

- two resistors in parallel (three ways, which one is omitted),
- two resistors in parallel and the third one in series to them (three ways),
- two resistors in series and the third one parallel to them (three ways)
- and finally all three resistors in parallel (a single way).

In order not to destroy the fuse, the total resistance must be at least $\frac{230\ \mathrm{V}}{15\ \mathrm{A}} \doteq 15.5\ \Omega$. The scheme without resistors will short-circuit the heater and blow the fuse. The scheme with three resistors in parallel will only produce total resistance

$$R = \frac{1}{\frac{1}{20\ \Omega} + \frac{1}{30\ \Omega} + \frac{1}{60\ \Omega}} = 10\ \Omega,$$

which is too little and destroys the fuse as well. The same is true for resistors $20 and 30\ \Omega$; and also $20 and 60\ \Omega$ when connected in parallel – their resistances are only $12\ \Omega$ and $15\ \Omega$ respectively. All other schemes are safe.

So we are left with 14 schemes. However, two of them have the same resistance: if we connected the second and third resistor in parallel, we obtain $R = 20\ \Omega$, which is equal to using only the first resistor. Therefore there are only 13 possible power settings of Luke's heater.

18 As the hands are moving with constant average angular speeds and pass the XII mark simultaneously, there will be two similar solutions, differing only in the direction of the torque; and both will be equally far away from the midnight or noon. We will only analyse the solution where the force acts anticlockwise, which occurs in the evening. The morning solution will be its mirror image with respect to the vertical axis.

At first we need to realize that the minute hand is twice as long as the hour hand, but its mass is only half as much, so both hands exert the same torque as long as the angles are equal. The time needed to complete one revolution is 43200 seconds for the hour hands and 3600 seconds for the minute hand. We may express the acting torques in terms of angles (measured from the XII mark)

$$M = M_{\mathrm{hour}} + M_{\mathrm{min}} = 2md \sin\left(\frac{2\pi t}{43\,200\ \mathrm{s}}\right) + 2md \sin\left(\frac{2\pi t}{3600\ \mathrm{s}}\right). \tag{18.1}$$

To find the maximum, we may differentiate this function. If we are not able or willing to do that, it is reasonably simple to think a bit and then use a calculator to find the solution by trial and error. Also values of m and d are constant, and since we only want to know the time and not the magnitude of the torque, we may ignore them altogether.

We should intuitively understand than the hands will exert maximum torque at about quarter to nine, when both hands are close to the IX mark. However, at 08:45 exactly the hour hand is still 3.75° below the mark and its torque is still increasing with time. The exact maximum will thus occur slightly later.

For the exact second 08:45:00 we substitute $t = 31\,500$ s to 18.1 and using a calculator we find out that

$$M_{08:45:00} \doteq 1.991\,445 \cdot 2mg.$$

By trial and error we do the same for next few seconds. Soon we will find out that the torque increases slightly a few more times, but after $t = 31\,506\ \mathrm{s} = 08{:}45{:}06$ it starts to decrease again. Other local maxima between 06:00 a 12:00 are necessarily smaller.

Finally we may want to find the mirrored solution, or the complement to 43200 seconds. So the solutions are

$$t = 11\,694\ \mathrm{s} \equiv 03{:}14{:}54, \qquad \text{and} \qquad t = 31\,506\ \mathrm{s} \equiv 08{:}45{:}06,$$

or, in both cases, exactly twelve hours later.

$\boxed{\mathbf{19}}$ A cube immersed into a liquid follows the Archimedes' principle. Its straightforward consequence is that a ratio of densities of the cube and the liquid is equal to a ratio of an immersed volume of the cube to the volume of the entire cube. Let a be a length of cube side, ρ_w a density of water and ρ_c a density of the cube, then in the first situation,

$$\frac{\rho_c}{\rho_w} = 1 - \frac{a^2 \cdot 4\ \mathrm{cm}}{a^3}. \tag{19.1}$$

In the second situation, the a height of the emerged part is 18 cm. It has got a shape of a regular pyramide, the upper edges of which form right angles.

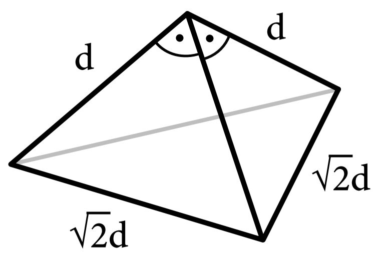


Figure 19.1: *An emerged part of the cube*

Denote a length of these edges d. A volume of the pyramide is $\frac{1}{6}d^3$. Based on the Pytagorean theorem, edges along a base of the pyramid are $\sqrt{2}d$ long. Let's focus on the triangular base of the pyramide.

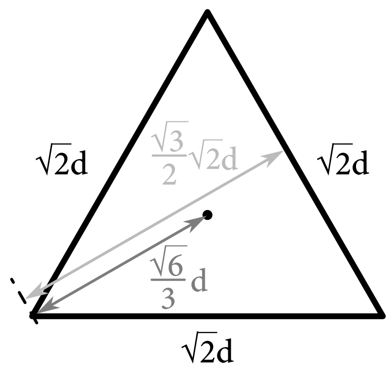


Figure 19.2: *The intersection of the cube and a water surface*

It has a shape of an equilateral triangle with an altitude $\frac{\sqrt{6}}{2}d$ long. A distance of a centre of the based from its vertices is $\frac{\sqrt{6}}{3}d$. Finally, consider a right-angled triangle determined by the centre of the base, a vertex of the base and the upper vertex of the pyramide.

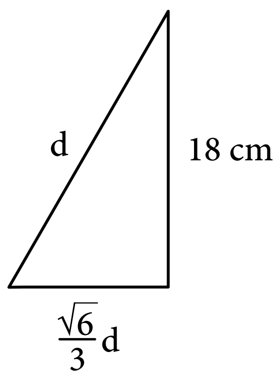


Figure 19.3: *The considered right-angled triangle*

The Pythagorean theorem gives $d = 18\sqrt{3}$ cm. In the second situation,

$$\frac{\rho_c}{\rho_w} = 1 - \frac{\sqrt{3}\cdot(18\text{ cm})^3}{2a^3}, \tag{19.2}$$

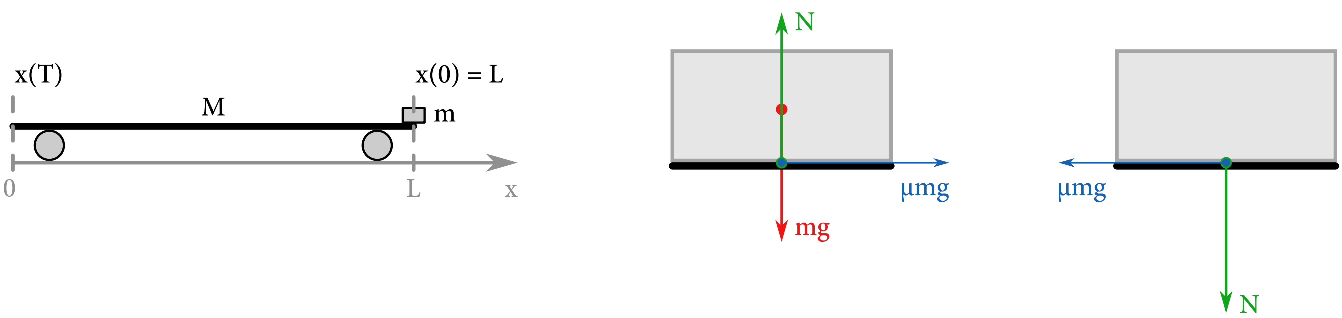
and comparing 19.1 and 19.2 yields

$$a = 3^{\frac{13}{4}}\text{ cm}.$$

Finally, substituting into 19.1 we get

$$\rho_c = \left(1 - 4\cdot 3^{-\frac{13}{4}}\right)\rho_w \doteq 887\text{ kg/m}^3.$$

20 Shortly after accelerating the skateboard, the stone is still in rest at the front end of the skateboard. The skateboard accelerated to an initial velocity **v** starts to decelerate immediately due to a friction force between the stone and the skateboard. As a consequence of Newton's third law, the stone starts to accelerate. The situation is depicted on the picture:



Firstly, we find accelerations of the stone **a** and of the skateboard **A**:

$$a = \mu g, \qquad A = -\mu\frac{m}{M}g. \tag{20.1}$$

Then we can write equations for position and velocity of the stone with an initial position L

$$x_K(t) = L + \frac{1}{2}\mu g t^2, \tag{20.2}$$

$$v_K(t) = \mu g t \tag{20.3}$$

and of the skateboard with an initial position 0

$$x_S(t) = vt - \frac{1}{2}\mu g \frac{m}{M} t^2, \tag{20.4}$$

$$v_S(t) = v - \mu \frac{m}{M} g t. \tag{20.5}$$

In order the stone not to fall down of the skateboard, it has to stop moving with respect to the skateboard by it passes distance L. Denote the entire time since the start of moving to an equalisation of velocities as τ. We can state following conditions for position and velocity

$$x_S(\tau) = x_K(\tau). \tag{20.6}$$

$$v_S(\tau) = v_K(\tau), \tag{20.7}$$

After applying of 20.5 and 20.3, the condition for velocity 20.7 yields an equation for τ:

$$v - \mu \frac{m}{M} g\tau = \mu g \tau, \tag{20.8}$$

therefrom

$$\tau = \frac{v}{\mu g} \frac{M}{m+M}. \tag{20.9}$$

Analogiously, after substituting of 20.4 and 20.2 into the condition for position 20.6, we obtain

$$L + \frac{1}{2}\mu g \tau^2 = v\tau - \frac{1}{2}\mu g \frac{m}{M}\tau^2.$$

Substituting for τ from 20.9 leads to

$$L = v\tau - \frac{1}{2}\mu g \frac{m+M}{M}\tau^2,$$

$$L = v\frac{v}{\mu g}\frac{M}{M+m} - \frac{1}{2}\mu g \frac{m+M}{M}\frac{v^2}{(\mu g)^2}\left(\frac{M}{M+m}\right)^2,$$

$$L = \frac{v^2}{\mu g}\frac{M}{m+M} - \frac{1}{2}\frac{v^2}{\mu g}\frac{M}{m+M}.$$

Therefrom

$$v = \sqrt{2\mu g L\left(\frac{m+M}{M}\right)},$$

where we have omitted a negative solution as being non-physical.

$\boxed{\mathbf{21}}$ Let's start with a simpler and more obvious condition: at the moment of landing when the height of a centre of mass above the ground is zero, the smartphone must be rotated by a multiple of 2π. The smartphone falls in homogeneous gravity field with gravity g from height h in time

$$t = \sqrt{\frac{2h}{g}}.$$

The smartphone on the table was laying on its back side, thus it has to be rotated by an angle $2\pi k$ in time t, where k is an arbitrary integer. Therefore

$$\omega = \frac{2\pi k}{\sqrt{\frac{2h}{g}}}.$$

The second relevant condition is that at the time of landing, upper and bottom edges of the smartphone must move downward. Otherwise, they would need to rise from a floor what is physically impossible and furthermore, it contradicts an assumption that landing is the first touch with the floor. A slower edge moves with velocity

$$v = -gt + \omega\frac{l}{2}\cos\varphi,$$

where φ is an angle between the smarthphone and the floor. At the moment of hitting the ground, it has to be 0°, thus a condition $v \leq 0$ leads to

$$\sqrt{2hg} = gt \geq \omega\frac{l}{2}.$$

We are interested in maximal allowed angular velocity ω of the smartphone, at which both conditions are still satisfied. Applying both conditions results in a condition

$$k \leq \frac{2h}{\pi l},$$

which yields the highest possible number of rotations $k = 3$ for given numeric values.

22 In order for the beam to return along the same trajectory as it came in, it has to be reflected perpendicularly from one of the the mirrors. Let's reverse the situation and trace the beam backwards, starting with the beam coming out of one of the mirrors at an angle of 90°. Firstly, let's assume, that the beam will be reflecting from the mirrors forever. Later on, it will become clear why it is useful.

For the first time the beam will hit the mirror at an angle $90° - \alpha$. Of course, it will reflect at the same angle. In the picture, there is a case, when the beam will after a few reflections hit one of the mirrors at an angle $90° - k\alpha$:

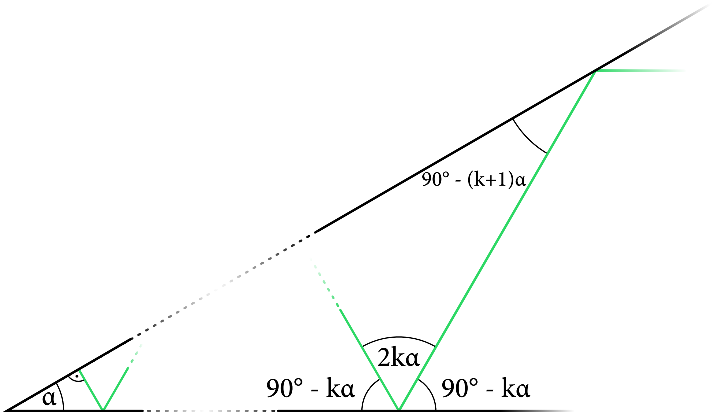


Figure 22.1: *The beam reflects until it reflects at an angle kα.*

The sum of the angles in the triangle implies that at the next reflection the beam hit the mirror at an angle $90° - (k+1)\,\alpha$. The beam will therefore reflect at the angles $90°, 90° - \alpha, 90° - 2\alpha, 90° - 3\alpha, \ldots$

The beam will leave the system in parallel to one of the mirrors only if one of those angles is equal to 0°. Therefore we are looking for angles α, which satisfies $90° - k\alpha = 0°$, where k is any natural number. The solutions are therefore all angles in the form $90°/k$.

$\boxed{23}$ First of all, we have to realise that bounces are purely elastic and hence the first bounce just turns the direction of the x-component of velocity of bouncy ball. Hence, we can imagine the wall as 'mirrors' as if the ball's motion is mirrored by the presence of walls. Let's imagine that there is no wall and the floor continues. Then, at the distance $2d$ from Charlie, there another Charlie who catches the ball right after its bounce from the floor:

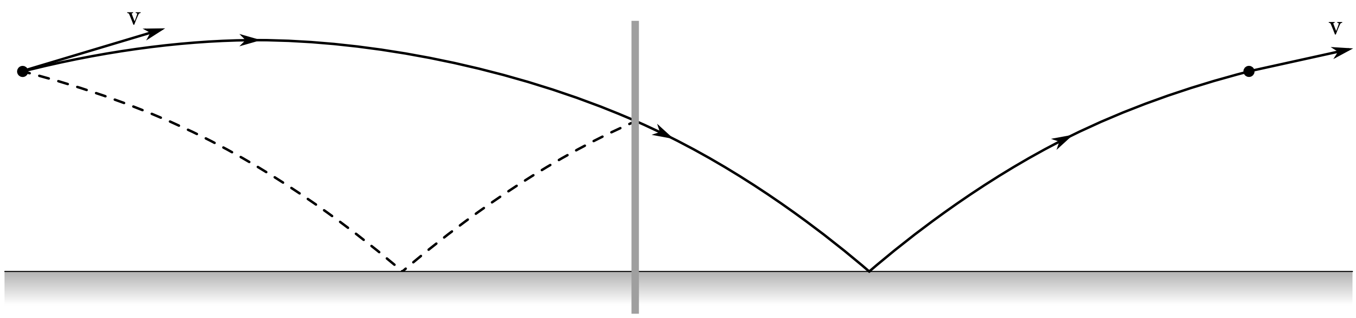


Figure 23.1: *Equivalent situation after 'mirroring'.*

Since the second bounce is also purely elastic, the y-component of velocity just turns it direction. Then, due to the law of conservation of energy it flies to the very same height as at the start, where it has the very same velocity as at the start. This situation is path-equivalent to the case when the ball is thrown from ground, up to the fact that same of the parabolic path is cut from its start and put on its end. Therefore, the time flight does not change also.

Let's consider bounce ball thrown at angle φ at velocity v_0 in a way that path covers distance $2d$ and at height h it has velocity v.

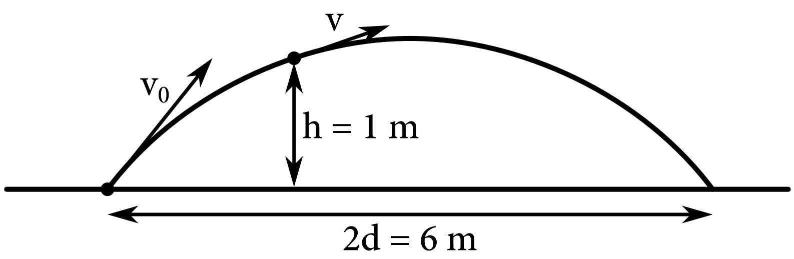


Figure 23.2: *Throw for which we want to calculate the angle.*

In order to calculate a time the ball needs to cover a parabolic path, we need its initial velocity v_0. This can be expressed from the law of conservation of energy,

$$v_0 = \sqrt{v^2 + 2gh}.$$

The vertical component of its velocity at the start is $v \sin \varphi$. That has to be also velocity at the end of trajectory. Hence, the time of flight can be expressed as

$$t = \frac{2v_0 \sin \varphi}{g}.$$

Unfortunately we do not know *a priori* an angle φ. The vertical distance covered by the bouncy ball is

$$2d = v_0 \cos \varphi t = \frac{2v_0^2 \sin \varphi \cos \varphi}{g} = \frac{v_0^2 \sin(2\varphi)}{g},$$

and thus

$$\varphi = \frac{1}{2} \arcsin\left(\frac{2dg}{v_0^2}\right).$$

From there

$$t = \frac{2\sqrt{v^2 + 2gh} \sin\left(\frac{1}{2} \arcsin \frac{2dg}{v^2+2gh}\right)}{g}.$$

After inserting the provided numerical values, we obtain $t \doteq 0.567$ s.

24 Let's name the nodes A, B, C, D, E.

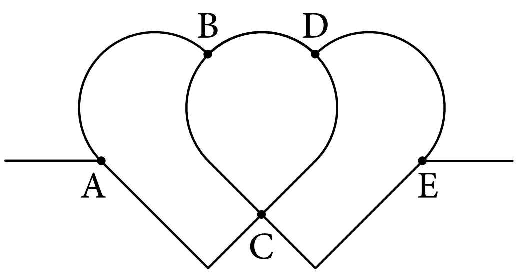
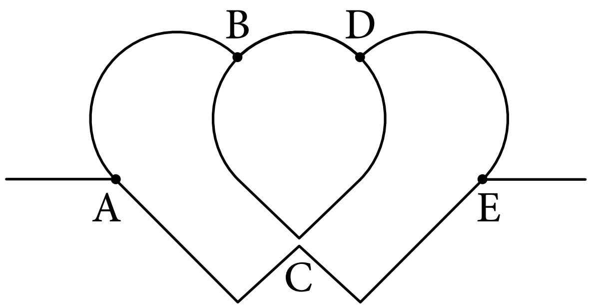


Figure 24.1: *The circuit with named important points*

Thanks to the symmetry the current between A and C (not through B), is the same as between the points C and E. Similarly the current between B and C is the same as between C and D. Therefore the circuit can be split into two at the point C:



The resistance between A and B and as well between D and E because of the symmetry is

$$R_{AB} = R_{DE} = \pi r\rho.$$

Between the points A and E the resistance is $6r\rho$. There are two parallel wires between the points B and D, one with the resistance of $\frac{1}{2}\pi r\rho$ and the second one with the resistance of $(2+\pi)\, r\rho$.

The resulting resistance between the points B and D therefore is

$$R_{BD} = \frac{2\pi + \pi^2}{3\pi + 4} r\rho.$$

And then the resistance between the points A and E through the upper wire is

$$\left(2\pi + \frac{2\pi + \pi^2}{3\pi + 4}\right) r\rho = \frac{7\pi^2 + 10\pi}{4 + 3\pi} r\rho$$

The final desired resistance of the whole circuit therefore is

$$\frac{42\pi^2 + 60\pi}{24 + 28\pi + 7\pi^2} r\rho \doteq 3.3306 r\rho.$$

$\boxed{25}$ Denote height of a barrel H. The barrel is half-full, thus a volume of the air inside is $V_0 = \frac{H}{2}S$. A pressure in the barrel is equal to the atmospheric pressure. A pouring of the tea stops at the moment when a sum of the air pressure inside and a hydrostatic pressure of the tea is equal to the atmospheric pressure outside. Therefore,

$$p_a = p + \left(\frac{H}{2} - h\right)\rho g,$$

where h is a descent of a tea surface and ρ is a density of the tea.

A temperature of the air inside after stopping pouring of the tea gradually reaches the temperature of the air before pouring. According to the ideal gas low

$$p_0 \frac{H}{2} S = \left[p_a - \left(\frac{H}{2} - h\right)\rho g\right]\left(\frac{H}{2} + h\right) S.$$

Having rearanged the last equation, we obtain a quadratic equation for h

$$h^2 + \frac{p_a}{\rho g}h - \frac{H^2}{4} = 0.$$

Its positive solution is a sought descent of the tea surface

$$h = -\frac{p_a}{2\rho g} + \sqrt{\left(\frac{p_a}{2\rho g}\right)^2 + \frac{H^2}{4}}.$$

For $H = 1$ m, we obtain $h \doteq 25$ mm.

26 We will find an elastic energy of springs in the frame. An energy of a spring with stiffness k, elongated by x is

$$E = \frac{1}{2}kx^2.$$

Suppose, that an angle between two adjecent sides of the frame is α. Then one spring has length $x_1 = 2d\sin\frac{\alpha}{2}$ and the other one length $x_2 = 2d\cos\frac{\alpha}{2}$. Therefore, the overal energy of the frame is

$$E = 2kd^2\left(\sin^2\frac{\alpha}{2} + \cos^2\frac{\alpha}{2}\right) = 2kd^2.$$

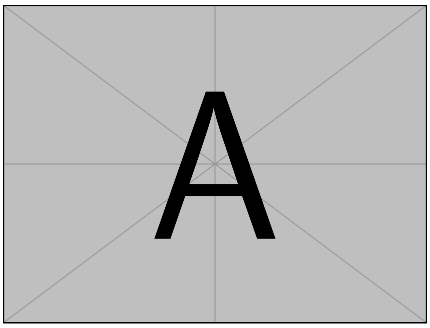
We see that the energy is independent of the angle α. It means that the force, which is necessary to be exerted in order to maintain the angle α, is zero.

27 We solve this problems utilising the law of conservation of energy. The sheep has got only a potential energy at the highest point of its path – at height h above flat ground. The potential energy is mgh_T, where m is sheep's mass and g is gravity. The rolling sheep has got both translational and rotational kinetic energy. An angular frequency of the rolling sheep is v/r and a moment of inertia of a homogeneous cylinder is $\frac{1}{2}mr^2$. Therefore, the total kinetic energy of the sheep is

$$\frac{1}{2}mv^2 + \frac{1}{4}\frac{mr^2v^2}{r^2} = \frac{3}{4}mv^2.$$

From the law of conservation of energy, we obtain

$$h_T = \frac{3v^2}{4g}.$$



Notice that the sheep does not touch the hill with its lowest point but with bit higher. Namely, its lowest point is r below its axis and a point of contact is $r\cos\alpha$ below the axis. Therefore, the grass on the hill is rolled over upto height

$$h = \frac{3v^2}{4g} + r\left(1 - \cos\alpha\right).$$

28 From symmetry we can see that each blade (triangle part) contributes to total moment of inertia the same. So, we want to express a moment of inertia just for one blade and multiply it by three. Besides that we can notice that from two blades we can build a square with side a.

We want to know the moment of inertia around perpendicular axis, which goes through one of the apexes. The moment of inertia around axis going through the center is found in tables,

$$I_\square = \frac{1}{6} m_\square a^2,$$

if we don't have the tables, we can calculate it by using scaling and Steiner's formula. This also help us to calculate the moment of inertia around the axis going through the apex.

If we mark the distance of new axis from the center of gravity $x = \frac{\sqrt{2}}{2}a$, we will get

$$I_{\square'} = I_\square + m_\square x^2 = \frac{1}{6} m_\square a^2 + m_\square \left(\frac{\sqrt{2}}{2} a^2\right) = \frac{2}{3} m_\square a^2.$$

Stella's propeller is made from three halves of stated square, so its moment of inertia will be 3/2 times bigger, so $m_\square a^2$. After using $m_\square = \sigma a^2$, we get result

$$I_\text{propeller} = \sigma a^4.$$

Other way of solving this problem is to realize that moment of inertia depends only on the perpendicular distance of points of the body from the rotation axis, and this axis in our propeller's case will not change, if we arbitrarily rotate the blades around the rotation axis. This means that we can rotate them in a way that we get a shape, which represents 3/8 of square with a side of legth $2a$. The propeller's moment of inertia is then 3/8 of moment of inertia of given square around its center, so

$$I_\text{propeller} = \frac{3}{8} \cdot \frac{1}{6} \cdot (2a)^2 \, \sigma \cdot (2a)^2 = \sigma a^4.$$

Moment of inertia is usually given in form constant times mass times characteristic dimension squared. Mass of the whole propeller is $M = \frac{3}{2}\sigma a^2$. If we use the length of the triangle's leg a as a characterictic dimension, the moment of inertia will be

$$I_\text{propeller} = \frac{2}{3} M a^2.$$

29 Imagine a spring made of n the same springs with stiffness k. If we act on this composed spring by certain force F, each of the springs shorten the same as if we act on the composed spring itself. Change in length will be n times higher and total stiffness of composed spring will be only $\frac{k}{n}$.

In terms of cutting the spring, it will behave differently – if we cut the spring into n pieces, and act by force F, each will shorten just by one n-tuple of the length (by which the original spring would shorten), what responds to nk. Alike, when we compose n the same springs with stiffness k next to each other, stiffness of the resultant spring will be nk. Because of that, the total stiffness of spring's pieces will be kn^2, if they are placed next to each other.

We have to realize that the rest length of those springs also decreased on a value $\frac{1\,\mathrm{m}}{n}$. In order to hold the wooden platform above the ground, the springs must be compressed by less than is their rest length. Force which is caused by wooden platform with Horacio and acts on the springs, is equal $F_S = g \cdot 110\,\mathrm{kg}$.

Spring force is $F_p = k'x = kn^2\Delta x$, where Δx is change in length of the spring. The condition for holding the wooden platform above the ground is

$$F_S < n \cdot 70\,\mathrm{N}.$$

It is easy to finish the calculations and get a result that Horacio has to cut the spring into at least 16 pieces.

30 Vo výsledku chceme zistiť osvetlený priestorový uhol. Ten je rovný podielu plochy gule, ktorú by sme osvietili, ak by sme našu kocku dali do stredu tejto gule a jej polomeru na druhú. Najjednoduchšie si to predstavíme tak, že si zoberieme guľu, ktorá má stred v strede kocky s priemerom dlhým ako stenová uhlopriečka kocky.

Ak si označíme dĺžku strany kocky a, táto guľa má polomer $\frac{\sqrt{2}}{2}a$. Navyše pekne vidíme, že táto guľa presne pretína obvody dier v bočných stenách. Ak by sme teraz kocku zmenšili, osvetlené by zostali tie časti gule, ktoré trčia z kocky a sú na bokoch (diery má lampión len v bočných stenách).

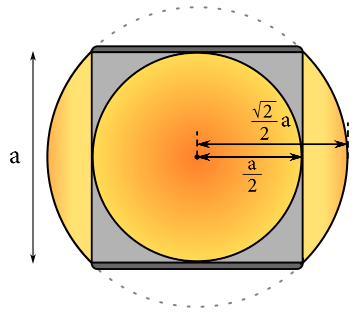


Figure 30.1: *Osvetlený priestorový uhol ako časť povrchu gule*

Aby sme vedeli zistiť akú plochu zaberajú osvetlené časti, musíme zistiť ich plochu a plochu gule. Tieto časti sú guľovými vrchlíkmi, o ktorých platí, že ich plocha je rovná $2\pi rh$, kde r je polomer gule, a h je výška vrchlíka. Všetky štyri osvetlené guľové vrchlíky budú mať zo symetrie rovnakú plochu. Ich výška bude rozdiel medzi polomerom gule a polovicou strany kocky, čiže

$$\frac{\sqrt{2}}{2}a - \frac{a}{2}.$$

Každý z nich bude mať plochu

$$2\pi\left(\frac{\sqrt{2}}{2} - \frac{1}{2}\right)\frac{\sqrt{2}}{2}a^2$$

a celková plocha je teda

$$8\pi\left(\frac{\sqrt{2}}{2} - \frac{1}{2}\right)\frac{\sqrt{2}}{2}a^2.$$

Ak to vydelíme štvorcom polomeru gule, dostávame príslušný priestorový uhol

$$8\pi\left(1-\frac{\sqrt{2}}{2}\right) \doteq 7.361\ \text{sr}.$$

$\boxed{31}$ A state of the gas is described by so-called state quantities. Commonly used state quantities are pressure p, volume V and temperature T. State variables has to satisfy an equation of state

$$pV = Nk_BT,$$

where N is number of gas particles and k_B is Boltzmann constant. It means that if we know two state quantities, the third one can be calculated.

If only forces that can act on piston are a force due to a spring and a force due to a pressure inside, the pressure inside satisfies $p = \frac{kh}{S}$, where k is a stiffness of the spring, h is its elongation and S is an area of the piston. It means that the elongation of the spring uniqly describes gas pressure, therefore it is also a good state quantity, as well as it describes a volume of the gas $V = Sh$. Inserting these expresions into the state equation yields

$$kh^2 = Nk_BT.$$

Realise that the elongation of the spring describes both a pressure and a volume. It means that if we know a volume, we can immediately calculate a pressure (and vice versa), therefore there is only one independent state quantity.

Oliver drew an h–T diagram of an adiabatic process with an ideal gas. An adiapatic process follows an equation

$$pV^{\varkappa} = \text{const}.$$

In our case, we obtain

$$kS^{\varkappa-1}h^{\varkappa+1} = \text{const}.$$

It means that if we deal with the adiabatic process, $h = \text{const}$. However, if there are no external forces acting on the system, the elongation of the spring will not alter. Thus, neither the pressure nor the volume will alter, and hence neither the temperature can alter. It means that if we are to consider the adiabatic process, e. i. if we forbid a heat transfer with surrounding, the gas state will not alter, therefore the adiabatic process will be represented by a single point on the h–T diagram.

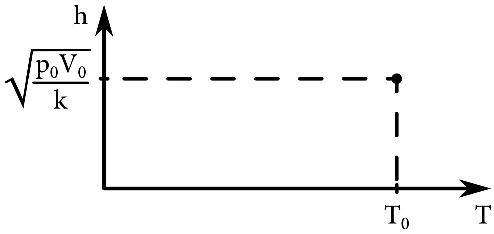
Let's find coordinates of this point. The state of the gas is given by state quantities p_0, V_0, T_0. Hence, the temerature is given, thus we need to find only the elongation of the spring h_0. Multiplying equations for pressure and for volume yields

$$p_0V_0 = \frac{kh_0}{S}\cdot Sh_0 = kh_0^2,$$

therefrom

$$h_0 = \sqrt{\frac{p_0V_0}{k}}.$$

Therefore, the sought diagram of the adiabatic process looks like this:



$\boxed{32}$ We will consider light being a transverse electromagnetic wave. Suppose, there is a light polarised in a certain direction. We let it pass through an ideal polarising filter with a plane of polarisation rotated with respect the direction of polarisation of the light by an angle φ. Behind the filter, we observe only a component in the direction determined by orientation of the plane of polarisation of the filter with an amplitude $A_0 \cos\varphi$. A perpendicular component with a magnitude $A_0 \sin\varphi$ does not pass through the filter.

Energy of an electromagnetic wave is equal to a squared amplitude, hence in terms of intensities, a portion with a magnitude $I_0 \cos^2\varphi$ passes, whilst a portion with a magnitude $I_0 \sin^2\varphi$ is absorbed by the filter. A sum of transmitted and absorbed energy is, of course, equal to the energy of entering wave.

If Simon wants horizontaly polarised light, the last filter has to be oriented horizontaly. What about other filters? Intuitively, we guess that they should be oriented in such a way that angles between planes of polarisation of two neighbouring filters were same and as little as possible. Let's prove it.

Consider three filters with orientations described by angles φ_1, φ_2 and φ_3. Differences between angles of their orientation differs from each other,

$$\beta \equiv \varphi_3 - \varphi_2 \neq \varphi_2 - \varphi_1 \equiv \alpha.$$

We will show that if we rotate the middle filter in such a way that the differences are equal, a resulting intensity will increase. If we rotate only the middle filter, we have

$$\alpha + \beta = \gamma = \text{const} \qquad \Rightarrow \qquad \beta = \gamma - \alpha$$

thus the resulting intensity after passing the filters is

$$I_{\text{behind}} = I_{\text{in front}} \cdot \cos^2\alpha \cdot \cos^2(\gamma - \alpha)\,.$$

By means of differential calculus or by a different method, we find out that this function reaches its maximum value when $\alpha = \gamma/2$, respectively $\alpha = \beta$. Rotating of the middle filter does not affect the direction of polarisation behind the third filter, therefore always, when we find such a triple of filters, we can optimese it by rotation of the middle filter. The only layout that cannot be optimesed is the one in which the differences between angles of orientations of the neighbouring filters are equal.

It also proves that Simon has to use all filters. If he ommited one of them, it would be same as if he placed it in front of the first one with same orientation. However, we already know that such a layout could be optimised by rotating of the second filter, therefore this layout cannot be optimal.

Finally, we need to calculate how much light passes through the optimal layout of the filters. Each of n filters rotates the plane of polarisation by angle $90°/n$ and lets pass only $\cos^2(90°/n)$ of intensity of the light. Each filter lowers the intensity by same factor, therefore the resulting intensity for $n = 10$ is

$$I = I_0 \cos^{20}\left(\frac{90°}{10}\right) \doteq 78.05\ \%.$$

33 An explosion of a black powder supplied the gas with an energy E. Provided the explosion was sufficiently short, so that a projectile had no time to move, the gas did not do any work. Therefore all the supplied heat resulted to an increase of an internal energy of the gas. It lead to an increase of a temperature of the gas given by an equation $E = \gamma Nk\Delta T$, wher N is a number of particles of the gas, k is Boltzmann constant and γ is a constant reflecting a number of degrees of freedom of gas molecules. For two-atomic molecules $\gamma = \frac{5}{2}$.

If an original temperature was T_0, a new temperature after the explosion was $T_1 = T_0 + \Delta T$. The explosion was an isochoric process, therefore a pressure after the explosion was

$$p_1 = p_0\left(1 + \frac{\Delta T}{T_0}\right).$$

Consequently, an aiabatic expansion occured. For adiabatic process, we have $pV^\varkappa = \text{const.}$, therefore $p_1 V_0^\varkappa = pV^\varkappa$. It yields

$$p(V) = p_0\left(1 + \frac{\Delta T}{T_0}\right)\left(\frac{V_0}{V}\right)^\varkappa.$$

The work done by the gas can be calculated by integrating

$$W = \int_{V_0}^{LS} p_0\left(1 + \frac{\Delta T}{T_0}\right)\left(\frac{V_0}{V}\right)^\varkappa \mathrm{d}V.$$

However, we can avoid evaluating the integral if we realise that there is no heat flow between the gas and its surrounding during an adiabatic process. Therefore, according to the first low of thermodynamics, a work done by the gas is equal to a negative change of a gas internal energy $W = -\delta U$. In order to determine a change of internal energy, we nee to find a change of temperature of the gas during the adiabatic expansion. According to the ideal gas law

$$p_2 = p_1 \frac{T_2}{T_0 + \Delta T}\frac{V_0}{LS}.$$

Having inserted into the equation of adiabatic process, we obtain

$$p_1 V_0^\varkappa = p_1 \frac{T_2}{T_0 + \Delta T}\frac{V_0}{LS}(LS)^\varkappa,$$

from where

$$T_2 = \left(\frac{V_0}{LS}\right)^{\varkappa - 1}(T_0 + \Delta T).$$

Consequently, the sought change of internal energy is

$$\delta U = \gamma Nk\delta T = \gamma NK(T_0 + \Delta T)\left[\left(\frac{V_0}{LS}\right)^{\varkappa-1} - 1\right].$$

As $W = -\delta U$, we derive

$$W = (\gamma p_0 V_0 + E)\left[1 - \left(\frac{V_0}{LS}\right)^{\varkappa-1}\right].$$

The work done by gas is used for accelerating of the projectile with mass m. The projectile is provided with a kinetic energy $T = \frac{1}{2}mv^2$. Thus the sought velocity of the projectile is

$$v = \sqrt{\frac{2}{m}\left(\frac{5}{2}p_0 V_0 + E\right)\left[1 - \left(\frac{V_0}{LS}\right)^{2/5}\right]},$$

where we have used that $\gamma = \frac{5}{2}$ and $\varkappa = \frac{7}{5}$ for two-atomic gas.

34 Since we are only interested in luminous flux as viewed from a great distance and in the direction perpendicular to the shoreline, we only need to consider rays that exit the lanterns very close to this perpendicular line. Let's analyze a ray, exiting the lantern in a small angle α with respect to this line, for both lanterns.

With Nina's lantern it is easy – the ray simply travels in a straight line forever. With Hannah's lantern the ray also exits the lantern with angle α, but once it exits the glass cube the angle will be different. We may determine this angle using Snell's law

$$n_1 \sin\alpha_1 = n_2 \sin\alpha_2,$$

where n_1 and n_2 are indices of refraction of glass and air and α_1 and α_2 are angles in which the ray travel in and out of the cube.

Since angle α is very small, we may use the approximation $\sin\alpha \approx \alpha$. The index of refraction of air n_2 is very close to 1, and we know that $n_1 = n$. Hence $\alpha_2 \approx n\alpha$.

All light exiting Hannah's lantern with angles less than α from the sailor's line of sight forms a cone with apex angle 2α. After refraction at the face of the glass cube the apex angle of the new cone will be $2n\alpha$. The total luminous power stays constant, so the power density is inversely proportional to the area of the base of the cone. This is n^2 times larger for Hannah's light source, which mean the sailors will perceive Nina's lantern as n^2 times brighter.

35 A force acting on a moving charged particle with a velocity $\mathbf{v}$ and a charge q in a magnetic field with strength $\mathbf{B}$ is

$$\mathbf{F} = q\mathbf{v} \times \mathbf{B}.$$

In a considered case, the velocity of both particles is perpendicular to the magnetic field. Therefore, the force acting on the particles is a centripetal force. It can be expressed as

$$F_d = \frac{mv^2}{r},$$

thus

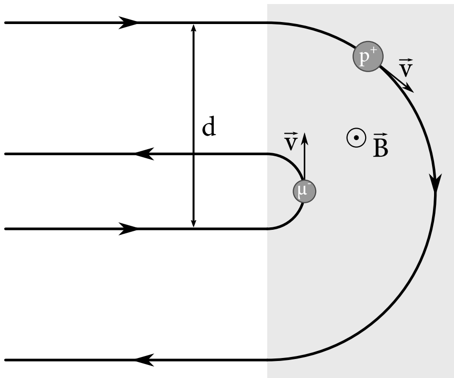
$$r = \frac{mv}{qB}.$$

Let's note that both particles move with relativistic velocities, therefore $m = \gamma m_0$, where m_0 is a rest mass of the particle and $\gamma = 1/\sqrt{1-\frac{v^2}{c^2}}$.

A muon charge differs from a proton charge only by a sign. In order to follow concentric half-circles inside the magnetic field, the particles had to entered the chamber at the mutual distance

$$d = r_e + r_p = \frac{v\left(m_p + m_\mu\right)}{eB\sqrt{1 - \frac{v^2}{c^2}}}.$$

For given numeric values $d = 2.44$ m.



36 Suppose that blankets have reached thermal equilibrium, thus an absorbed power is radiated back to the surrounding space entirely. Based on the symetry, we can conclude that energy is radiated equally to both sides of the blankets. Considering, the blankets are sufficiently close to each other, the entire power radiated by one blanket towards other one is completely absorbed by the other blanket.

Enumerate the blankets starting with the blanket closest to the Sun. Areal power densities F[2] of individual blankets have to satisfy following set of equations

$$F_1 = F_\odot + \frac{1}{2}F_2, \qquad F_2 = \frac{1}{2}F_1 + \frac{1}{2}F_3, \qquad F_3 = \frac{1}{2}F_2.$$

The system of equations has following solution:

$$F_1 = \frac{3}{2}F_\odot, \qquad F_2 = F_\odot, \qquad F_3 = \frac{1}{2}F_\odot.$$

We are interested only in the last equation.

Realise that we need to calculate an areal power density related to a unit surface area and it is only a half of F_3, as the blanket radiates only a half of its power in each direction. According to the problem statement, the blankets can be considered being ideal black bodies, therefore a temperature of the third blanket is

$$T = \sqrt[4]{\frac{F_\odot}{4\sigma}} \doteq 278.785\ \mathrm{K}.$$

37 Hydrostatic pressure increases with heigh as $p(h) = h\rho g$. A pressure in a liquid in a given point is independent of the orientation of surface, thus a magnitude of the force acting on a barrel at a height h is

$$\Delta F(h) = p(h) \cdot \Delta S = h\rho g \cdot 2\pi R \Delta h.$$

[2] F is the overal areal power density radiated by both sides of a blanket

This force acts along the entire circumference of the barrel at the heigh h in the perpendicular direction to the barrel surface.

A tension force due to that force in a stripe of the barrel with thickness Δh at the height h can be determined based on the principle of virtual work. Imagine that the force due to the pressure extends the radius of the barrel by δR. The corresponding virtual work done on the barrel is $\delta W = \Delta F \cdot \delta R$.

Exactly the same virtual work would be done by the tension force if it caused the same deformation. As the circumference is lengthened by $2\pi\delta R$, the virtual work done by tension force is $\delta W = \Delta T \cdot 2\pi\delta R$. An equality of virtual works yields $\Delta T(h) = \frac{\Delta F(h)}{2\pi}$. Therefore, the tension force at height h is

$$\Delta T(h) = R\rho g h \Delta h.$$

The tension force stretches the stripe of thickness Δh. However, we are interested in the force which stretches the entire barrel, thus we need to sum a contribution of all stripes over the entire height of the barrel. One option is to evaluate the integral

$$T = \int_0^H R\rho g h \, \mathrm{d}h.$$

The other one is to realise that the increment of the tension force depends linearly on the height, therefore it is represented by a streight line crossing the origin on the graph. The overal tension force is the area below the line,[3] what is nothing more than the area of a right-angled triangle with catheti of lengths H and $R\rho g H$, thus the overal tension force is

$$T = \frac{1}{2} R\rho g H^2.$$

One rim can withstand the tension force given by its ultimate tensile strength and its crossectional area $\tau = \sigma S$, therefore the minimum number of rims is

$$N = \left\lceil \frac{R\rho g H^2}{2\sigma S} \right\rceil.$$

For the given numerical values, we obtain $N = 11$.

38 Let's build on a solution of the previous problem about polarising filters. The only difference is that an intensity decreases not only due to different orientations of the filters but also due to lower translucency of the filters. We have to consider two counteracting effects – the more filters are used, the more economically the plane of polarisation can be rotated, but the higher are losses due to an absorption.

As a result, an additional exponential term appears in the formula for resulting intensity reflecting the loss of intensity when passing through each of the filters. We obtain

$$I = I_0 \cdot 0.9^n \cdot \left(\cos \frac{\pi}{2n}\right)^{2n}.$$

We examine a few first options (or alternativelly, we can use methods of differential calculus, what we do not recommend) and we find out that maximum is reached at $n = 5$,

$$I = I_0 \cdot 0.9^5 \cdot \cos\left(\frac{\pi}{10}\right)^{10} \doteq 35.75\ \%.$$

[3] Similarly to uniformly accelerated motion, where the increment of distance is equal to the area below the graph $v(t)$.

39 Let's consider a small area of a soup bubble ΔS. We will start with a calculation of a force acting on this area due to a surface tension. The surface tension of the bubble results in a Laplace pressure p_c. The Laplace pressure is described by Young-Laplace equation. The bubble has two surfaces, thus

$$p_c = \frac{4\gamma}{R},$$

where γ is a surface tension of soup watter and R is a radius of the bubble. Then a force acting on the area ΔS is

$$\Delta F_k = p_k \Delta S = \frac{4\gamma}{R} \Delta S.$$

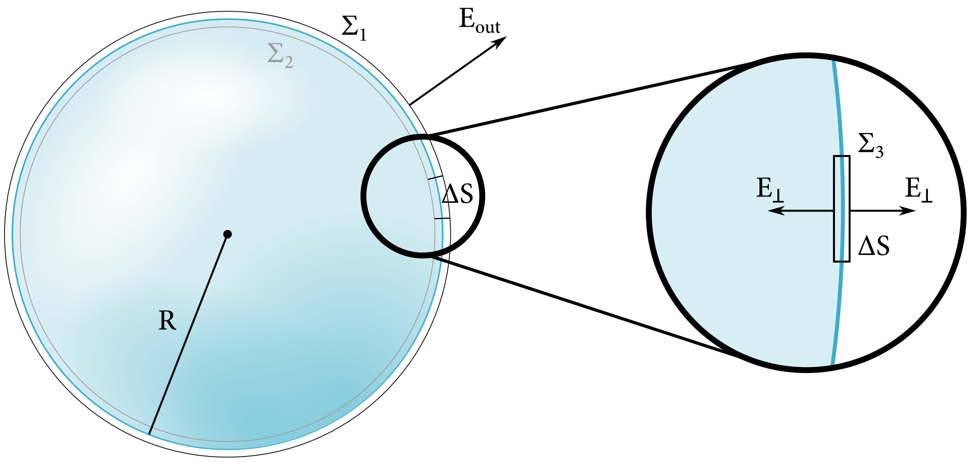
This force attracts the area to the bubble.

Secondly, we need to determine an electrostatic force which repels the area from the bubble after an electric charge has been brought onto the bubble. Assume that Mary has brought a charge Q. The bubble is conductive, therefore the charge has distributed uniformly over the entire outer surface of the bubble. An areal charge density on the bubble surface is

$$\sigma = \frac{Q}{4\pi R^2}.$$

A key tool for solving this problem is Gauss's law. It states that an integral of an electric intensity over an arbitrary closed surface is proportional to an overal electric charge inside the surface. In terms of math

$$\oint_S \mathbf{E} \cdot \mathrm{d}\mathbf{S} = \frac{q}{\varepsilon_0}.$$



A charge distribution on the bubble has got a spherical symmetry, hence the electric intensity outside the bubble has a spherical symmetry too. Consider a sphere Σ_1 at the outer surface of the bubble as a Gauss's surface. The intensity over this surface has got a radial direction and a constant magnitude. Therefore

$$\oint_{\Sigma_1} \mathbf{E} \cdot \mathrm{d}\mathbf{\Sigma_1} = E_{\text{out}} 4\pi R^2 \stackrel{!}{=} \frac{Q}{\varepsilon_0}.$$

Therefrom, the intensity at the bubble surface is

$$E_{\text{out}} = \frac{Q}{4\pi\varepsilon_0 R^2}.$$

Next, consider a sphere Σ_2 at the inner surface of the bubble as a Gauss's surface. In such a case, an overal charge inside the surface is zero and thus, based on the same arguments as in the previous situation, the electric intensity is

$$E_{\text{in}} = 0.$$

We have obtained all of this when we were looking at the bubble on a global scale. Now, let's look locally only at an area ΔS. If we are sufficiently close, we do not see a curvature of the bubble and the area seems to be flat. [4] Choose a Gauss's surface in a shape of a can Σ_3 with bases ΔS and with a tiny height. The can envelops the area ΔS of the bubble. As a charge distribution is planar, the electric intensity has got a reflection symmetry. The intensity has to be perpendicular to the area ΔS. If it were not, i. e. if it had also a tangential component with the area, it would cause an electric current flowing until the tangential component would be nullified by currents. According to Gauss's law

$$\oint_{\Sigma_3} \mathbf{E} \cdot \mathrm{d}\boldsymbol{\Sigma}_3 = 2E_\perp \Delta S \overset{!}{=} \frac{\sigma \Delta S}{\varepsilon_0},$$

from where

$$E_\perp = \frac{\sigma}{2\varepsilon_0} = \frac{Q}{8\pi\varepsilon_0 R^2}.$$

The intensity due to the area ΔS is non-zero at the inner surface of the bubble. However, a total intensity inside the bubble is zero, thus an intensity due to rest of the bubble at the inner surface of the area ΔS has to be $\mathbf{E} = -\mathbf{E}_\perp$. Consequently, a force repelling the area ΔS from the bubble is

$$\Delta F_e = E \Delta Q = e\sigma \Delta S = \frac{Q^2}{32\pi^2\varepsilon_0 R^4}\Delta S.$$

Finally, we can resolve the problem. If the pressure inside the bubble is to be equal to the atmospheric pressure, the attraction due to the surface tension has to be compensated by the electrostatic repulsion $\Delta F_c \overset{!}{=} \Delta F_e$. The stated equality yields

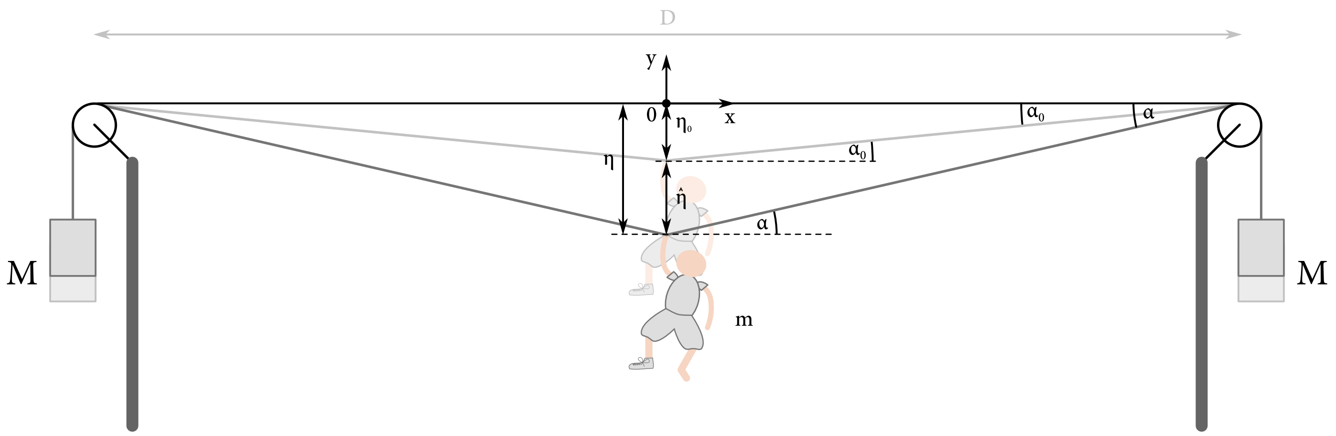
$$Q = \sqrt{128\pi^2\varepsilon_0 \gamma R^3}.$$

For given numerical values, we obtain $Q \doteq 1.31 \times 10^{-7}$ C.

40 Let's start with determination of a depth of the sag of the rope in equilibrium. Denote the depth of the rope sag as η_0. Let α be an inclination of the line. Then

$$\sin \alpha_0 = \frac{\eta_0}{\sqrt{\frac{D^2}{4} + \eta_0^2}}.$$

[4] Exactly due to a same reason why the Flat Earthers believe that the Earth is flat.



The rope is stretched by a tensile force $T = Mg$. A balance of forces in a vertical direction yields

$$2Mg\sin\alpha_0 = mg.$$

Having substituted for $\sin\alpha_0$, we obtain

$$\frac{\eta_0}{\sqrt{\frac{D^2}{4}+\eta_0^2}} = \frac{m}{2M}.$$

For future use, we express the last equation in the following forms:

$$\eta_0 = \frac{mD}{2\sqrt{4M^2-m^2}};$$

$$\sqrt{\frac{D^2}{4}+\eta_0^2} = \frac{MD}{\sqrt{4M^2-m^2}}.$$

Now, let η denote an actual sag of the rope in a dynamical case. Introduce Cartesian coordinate system, as illustrated on the picture. Assume, that a length of the entire rope is L. In that case, we can find position vectors of individual objects. A position vector of the left mass is

$$\mathbf{r}_\mathrm{L} = \left(-\frac{D}{2};\sqrt{\frac{D^2}{4}+\eta^2}-\frac{L}{2}\right),$$

of the right mass

$$\mathbf{r}_\mathrm{R} = \left(+\frac{D}{2};\sqrt{\frac{D^2}{4}+\eta^2}-\frac{L}{2}\right)$$

and of Jacob

$$\mathbf{r} = (0;-\eta)\,.$$

Next, we find velocities of individual objects. Denote vertical component of Jacob's velocity as $\dot\eta$. Then, vertical components of velocities of masses are $\dot\eta\sin\alpha$, which can be obtained from the geometry. Therefore, the sought velocities are

$$\begin{aligned}
\mathbf{v_L} &= \left(0; \frac{\eta\dot{\eta}}{\sqrt{\frac{D^2}{4}+\eta^2}}\right); \\
\mathbf{v_P} &= \left(0; \frac{\eta\dot{\eta}}{\sqrt{\frac{D^2}{4}+\eta^2}}\right); \\
\mathbf{v} &= \left(0; -\dot{\eta}\right).
\end{aligned}$$

Finally, we are able to determine an overal potential energy of the system

$$U = Mgy_L + Mgy_R + mgy = 2Mg\left(\sqrt{\frac{D^2}{4}+\eta^2} - \frac{L}{2}\right) - mg\eta$$

and its kinetic energy

$$T = \frac{1}{2}Mv_L^2 + \frac{1}{2}Mv_R^2 + \frac{1}{2}mv^2 = \frac{1}{2}\left(\frac{2M\eta^2}{\frac{D^2}{4}+\eta^2} + m\right)\dot{\eta}^2.$$

We are interested in small oscillations of Jacob around his equilibrium state. Hence, express the sag of the rope in a form

$$\eta = \eta_0 + \hat{\eta},$$

where $\hat{\eta}$ denotes an amplitude of the small oscillations.

In case of the small oscillations, an expression for the potential energy can be expanded to Taylor series. We want to investigate harmonic oscillations, therefore we need an expansion upto the second order.

Let's find an expansion of a function

$$f(\hat{\eta}) = \sqrt{\frac{D^2}{4} + \left(\eta_0 + \hat{\eta}\right)^2}$$

around zero. We need to know a value of the function $f(\hat{\eta})$ and of its first two derivatives for that. Gradually, we obtain

$$\begin{aligned}
f(\hat{\eta})\big|_{\hat{\eta}=0} &= \sqrt{\frac{D^2}{4}+\eta_0^2}; \\
\left.\frac{\mathrm{d}f(\hat{\eta})}{\mathrm{d}\hat{\eta}}\right|_{\hat{\eta}=0} &= \left.\frac{\eta_0+\hat{\eta}}{\sqrt{\frac{D^2}{4}+\left(\eta_0+\hat{\eta}\right)^2}}\right|_{\hat{\eta}=0} = \frac{\eta_0}{\sqrt{\frac{D^2}{4}+\eta_0^2}}; \\
\left.\frac{\mathrm{d}^2f(\hat{\eta})}{\mathrm{d}\hat{\eta}^2}\right|_{\hat{\eta}=0} &= \left.\frac{D^2}{4\sqrt{\frac{D^2}{4}+\left(\eta_0+\hat{\eta}\right)^2}^3}\right|_{\hat{\eta}=0} = \frac{D^2}{4\sqrt{\frac{D^2}{4}+\eta_0^2}^3}.
\end{aligned}$$

Therefore,

$$
\begin{aligned}
f(\hat\eta) &\approx f(\hat\eta)|_{\hat\eta=0} + \left.\frac{\mathrm{d}f(\hat\eta)}{\mathrm{d}\hat\eta}\right|_{\hat\eta=0}\cdot\hat\eta + \frac{1}{2}\left.\frac{\mathrm{d}^2f(\hat\eta)}{\mathrm{d}\hat\eta^2}\right|_{\hat\eta=0}\cdot\hat\eta^2 = \\
&= \sqrt{\frac{D^2}{4}+\eta_0^2} + \frac{\eta_0}{\sqrt{\frac{D^2}{4}+\eta_0^2}}\hat\eta + \frac{D^2}{8\sqrt{\frac{D^2}{4}+\eta_0^2}^3}\hat\eta^2 = \\
&= \frac{MD}{\sqrt{4M^2-m^2}} + \frac{m}{2M}\hat\eta + \frac{D^2}{8}\frac{\sqrt{4M^2-m^2}^3}{M^3D^3}\hat\eta^2 = \\
&= \frac{D}{\sqrt{4-\left(\frac{m}{M}\right)^2}} + \frac{m}{2M}\hat\eta + \frac{\sqrt{4-\left(\frac{m}{M}\right)^2}^3}{8D}\hat\eta^2.
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
U &\approx 2Mg\left(\frac{D}{\sqrt{4-\left(\frac{m}{M}\right)^2}} + \frac{m}{2M}\hat\eta + \frac{\sqrt{4-\left(\frac{m}{M}\right)^2}^3}{8D}\hat\eta^2 - \frac{L}{2}\right) - mg\left(\frac{mD}{2\sqrt{4M^2-m^2}}+\hat\eta\right) = \\
&= \frac{gD}{\sqrt{4-\left(\frac{m}{M}\right)^2}}\left(2M-\frac{m^2}{2M}\right) - MgL + \frac{Mg}{4D}\sqrt{4-\left(\frac{m}{M}\right)^2}^3\hat\eta^2 = \\
&= \frac{1}{2}MgD\sqrt{4-\left(\frac{m}{M}\right)^2} - MgL + \frac{Mg}{4D}\sqrt{4-\left(\frac{m}{M}\right)^2}^3\hat\eta^2.
\end{aligned}
$$

We see that an expression for the potential energy contains only absolute term, which shifts equipotential surfaces by a constant value, tus it does not affect movement, and a quadratic term, which corresponds to a harmonic oscillations. Having applied an analogy with a potential energy of a spring $E_{\mathrm{pot}} = \frac{1}{2}kx^2$, we can talk about an effective „stiffness“ of the system

$$
k = \frac{Mg}{2D}\sqrt{4-\left(\frac{m}{M}\right)^2}^3.
$$

Now, let’s investigate the kinetic energy. It has a form of $E_{\mathrm{kin}} = \frac{1}{2}\mu u^2$, thus we can refer to an efective mass of the system in the equilibrium state

$$
\mu = \frac{2M\eta_0^2}{\frac{D^2}{4}+\eta_0^2} + m = 2M\left(\frac{m}{2M}\right)^2 + m = m\left(\frac{m}{2M}+1\right).
$$

An angular frequency of an oscillator with a stiffness k and mass μ is given by

$$
\omega = \sqrt{\frac{k}{\mu}}.
$$

After substituting of the expressions for k and μ and performing long disgusting arithmetics, we obtain

$$\omega = \sqrt[4]{(2M)^2 - m^2}\sqrt{\frac{2}{m} - \frac{1}{M}}\sqrt{\frac{g}{D}}.$$

Having made a use of $\frac{2M}{m} = \frac{41}{9}$, the results simplifies to

$$\omega = \frac{16}{3}\sqrt{\frac{10}{41}}\sqrt{\frac{g}{D}}.$$

Finally, the sought period is

$$T = \frac{2\pi}{\omega} = \frac{3\pi}{8}\sqrt{\frac{41}{10}}\sqrt{\frac{D}{g}}$$

and for $D = 25$ m,

$$T = \frac{3\sqrt{41}}{16}\pi \text{ s}.$$

Answers

$\boxed{\mathbf{1}}$ 147 s

$\boxed{\mathbf{2}}$

$\boxed{\mathbf{3}}$

$\boxed{\mathbf{4}}$

$\boxed{\mathbf{5}}$

$\boxed{\mathbf{6}}$

$\boxed{\mathbf{7}}$

$\boxed{\mathbf{8}}$ 10 cm

$\boxed{\mathbf{9}}$

$\boxed{\mathbf{10}}$

$\boxed{\mathbf{11}}$

$\boxed{\mathbf{12}}$

$\boxed{\mathbf{13}}$

$\boxed{\mathbf{14}}$

$\boxed{\mathbf{15}}$

$\boxed{\mathbf{16}}$

$\boxed{\mathbf{17}}$

$\boxed{\mathbf{18}}$

$\boxed{\mathbf{19}}$

$\boxed{\mathbf{20}}$

$\boxed{\mathbf{21}}$

$\boxed{\mathbf{22}}$

$\boxed{\mathbf{23}}$ 2.116 s

$\boxed{\mathbf{24}}$

$\boxed{\mathbf{25}}$

$\boxed{\mathbf{26}}$

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$\boxed{\mathbf{29}}$

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