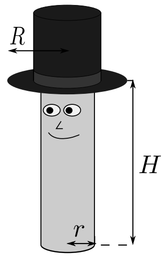
Problems

1. Jimmy has a water-proof top hat with radius R. Jimmy is so slim he resembles a vertical cylinder with radius $r < R$ and height H. How fast can Jimmy walk in the rain so that he doesn't get wet? The rain falls straight down vertically with velocity u.

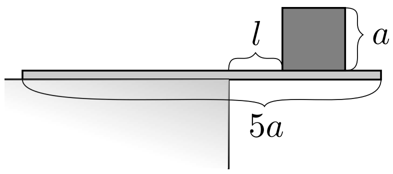


Obr. 1

2. The pressure in the water supply system is 20 atmospheres at the ground floor. What can the maximal height of a building be so that we can have a shower on the top floor?

3. How many times do we have to fold a sheet of paper to fill the whole distance from the Earth to the Sun? The thickness of the paper sheet is $100\,\mu\mathrm{m}$.

4. We are standing at the edge of a cliff. Everything we have is a cube of mass m and edge length a and a long plank of mass $m/2$ and length $5a$. What is the largest distance l between the edge of the cube from the edge of the cliff if it must not fall down?



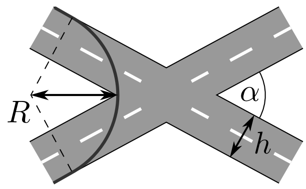
Obr. 2: Cube at the edge of the cliff

5. Martin would like to take a shower. But he's got a problem. Since he is spoiled, he needs water at least $T_{hot} = 30\,°\mathrm{C}$ hot and a flow rate of $Q = 0.1\,\mathrm{l\,s^{-1}}$. Unfortunately, he only has access to a water supply with water temperature $T_{cold} = 25\,°\mathrm{C}$. A through-flow heater seems to be the solution. What should be the electric energy consumption of the through-flow heater in order to satisfy Martin's needs? Density of water is $\rho = 1000\,\mathrm{kg\,m^{-3}}$, its specific heat capacity is $c = 4180\,\mathrm{J\,kg^{-1}\,K^{-1}}$ and the efficiency of the through-flow heater is close to $100\,\%$.

6. Submarines use ultrasound to measure the sea depth. A submarine emits ultrasound in all directions and waits for the echoes reflected from the sea floor. Let's consider a submarine moving horizontally with speed v. What is the depth under the submarine if the signal has returned after T seconds? The speed of sound in water is c.

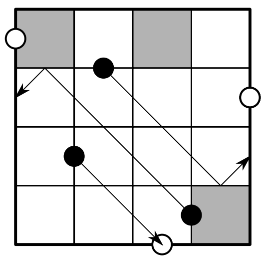
7. It was a hot day outside and Kate was about to cool her drink with an ice cube of edge length a. She put the drink with the ice cube on her scales and wanted to measure its weight. But Kate was irritated by the ice cube sticking out, so she pushed it below the surface. She immediately noticed the change of weight on the scales. How exactly had the weight changed? Densities of the drink and ice are ϱ_d and ϱ_i respectively.

8. Two roads of width h cross each other at an angle α. Frank the tractor driver would like to turn to the other road. What is the maximal radius of a circular trajectory which Frank can follow, if he may not run out from the road? Consider Frank and his tractor to be points.



Obr. 3

9. Who wants to solve a really difficult problem? Nobody? Well, let's play then. Three laser beams are emanating from black sources as drawn in the picture. The beams propagate in straight lines and they are reflected on surfaces of grey mirror cubes.
Where do we have to place the mirror cubes if we want the beams to hit the three white targets? We can only place the mirror cubes in the squares of the 4×4 grid. The outer walls do not reflect the beams.



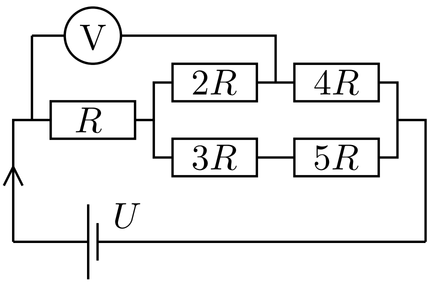
Obr. 4: Lasers and mirrors

10. Johnny has bought an enormous hot air balloon. Of course, his ambitions are just as huge, so he decided to fly around the Earth above the equator. Now he needs to know how much food he has to pack. How many days will the journey take? Round the result up to the next greater integer.
The mass of the balloon is 3000 kg, its diameter is 30 m, the coefficient of aerodynamic drag equals 0.16 and the temperature inside the balloon is 350 K. Consider an adiabatic atmosphere with standard conditions at the sea level 100 kPa and 15 °C. You may ignore the deformation of the balloon envelope. Wind at the equator blows eastward at a constant speed of $40\,\mathrm{km\,h^{-1}}$.

11. Jacob is definitely not a hockey player. Therefore he spends his time on ice just playing with various blocks. He has got two of them lying one on the other. The upper one's mass is $M = 4\,\mathrm{kg}$ and the bottom one's $m = 1\,\mathrm{kg}$. The coefficient of friction between these two blocks is $f = 0.32$. What will the acceleration of the upper block be if Jacob pushes the bottom block with a horizontal force of $F = 2\,\mathrm{N}$?

12. "George, wake up! You have to go to school!" "Mum, leave me alone. Do you know how difficult it is to walk there?" "No, do you?" Help him! Determine the amount of energy E needed to walk a distance s if the length of one step is l. You can assume that the energy is spent only for repeated lifting of the centre of gravity. George's body mass is m and the length of George's legs is h.

13. Julia gave George a voltmeter for his birthday. When he heard „Can you feel the voltage…" by Red Hot Chilli Peppers from the radio, he decided to measure the voltage in an electric circuit. The electric circuit is depicted below. What value will the voltmeter display when connected to the circuit as shown?



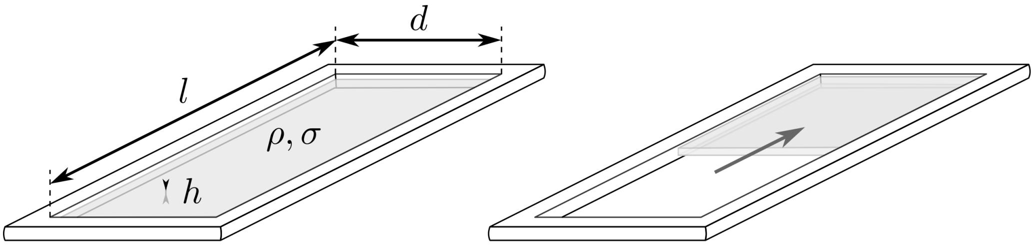
Obr. 5: George's electric circuit

14. A frog of mass 100 g has landed on the rim of a glass. The glass is a hollow conical frustum of negligible thickness and mass 50 g. The frog is now concerned about the stability of the glass. The radius of the rim is 5 cm and the radius of the base is 3 cm. How much water of density $1000\,\mathrm{kg\,m^{-3}}$ has to be poured into the glass so that it does not tip over?

15. We have a generator that burns solid fuel and works as follows: The fuel of mass m_0 with heat of combustion H is inserted into the generator and burnt. The efficiency of the generator is η. Water of mass m is used to draw the remaining energy converted to heat – it flows from a water tank filled with water of mass $M > m$ and cools the generator. Then it returns to the water tank and is mixed with the rest of the water. This cycle is repeated periodically. What is the temperature of water in the tank after n-th cycle if it was set to t_0 at the beginning and the specific heat capacity of water is c?

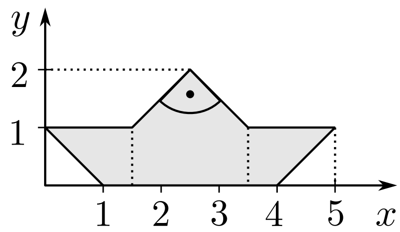
16. Irene played with a soap film of thickness $h = 1\,\mu\mathrm{m}$ and dimensions of $l = 10\,\mathrm{cm}$ and $d = 5\,\mathrm{cm}$ placed in a rectangular frame, as shown below. Suddenly the soap film burst along the shorter side of the rectangle and shrank to the other side. Irene wonders about the duration

of this process. Estimate the time of shrinking. The soap film is made of a soap solution with density $\rho = 1000\,\mathrm{kg\,m^{-3}}$ and surface tension $\sigma = 0.03\,\mathrm{N\,m^{-1}}$. Round the result to milliseconds.



Obr. 6: The burst of the soap film.

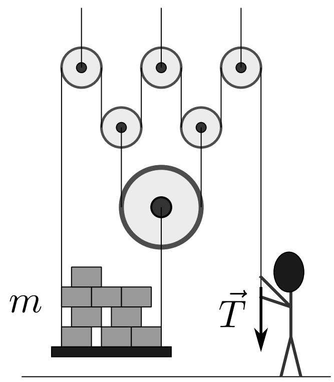
17. Johnny the pirate has finally saved enough money to buy a new ship. He went to a stationery store and bought one piece of paper. Then he proceeded to cut out a small ship as shown in the picture below. Now he wonders what the coordinates of its centre of gravity are.



Obr. 7: Overpriced luxurious boat

18. Billy the Kid is sitting on the floor of a large cave. He fires his Derringer straight up. The revolver's muzzle velocity is $200\,\mathrm{m\,s^{-1}}$. Five seconds later he hears the sound of the bullet hitting the ceiling. What is the height of the cave if the speed of sound is $330\,\mathrm{m\,s^{-1}}$?

19. "Why should I lift all these bricks by myself? No way! I had better built my own Atwood's machine, that would lift all the bricks instead of me. How much should I pull the rope if I want to lift pile of bricks of mass m?" Consider all of the ropes and pulleys to be nearly massless.



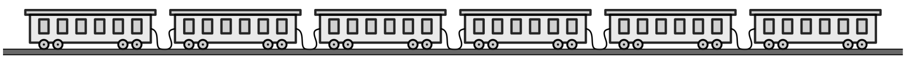
Obr. 8: Collosal Atwood's machine

20. In the practice of homeopathy, homeopaths use a so-called *centesimal* dilution: they take one drop of a solution of the active substance and dilute it in 99 drops of water. This is called a 1C dilution. To prepare a 2C dilution, they dilute 1 drop of a 1C solution in water and so on. The entire process is considered to be successfully finished when there are no molecules of the original substance left. How many times does a homeopath have to dilute the solution in order to obtain one teaspoon (5 ml) of the final product?

21. Toilet paper of length L and mass m is wrapped around a horizontal roll of radius r and negligible mass. The roll is placed on a horizontal holder. Suddenly a cat of mass M grabs the end of the toilet paper and the roll begins to unwind under the cat's weight. What is the angular velocity of the roll at the moment when the toilet paper is completely unwound? You may assume that $L \gg r$ and that there is no friction between the roll and the holder.

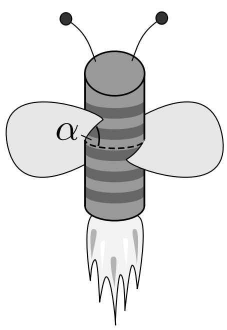
22. Sue wants to wash her hands, so she goes to a bathroom. After opening a pipe of radius $R = 2\,\mathrm{cm}$, water starts to flow at a rate of $Q = 0.2\,\mathrm{l\,s^{-1}}$. Sue notices that the jet of water narrows with increasing fall distance. Find the value of radius r of water jet at the basin floor placed $H = 33\,\mathrm{cm}$ under the pipe.

23. Michael used to play with trains when he was younger. He used to place N equally massive cars connected with perfectly inelastic ropes of length l. Then Michael placed the cars so close that all ropes were completely slack. Then he bumped the first car, so that it moved with velocity v_1. How much time passed before the last car started to move?



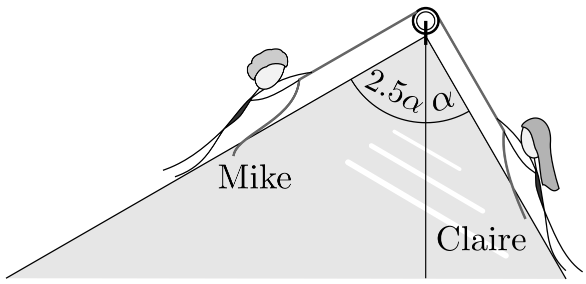
Obr. 9

24. Maya bought a small bee firework on New Year's Eve. The "bee" consists of a small rocket motor and two wings inclined to the horizontal plane at angle α. When the rocket engine is ignited, the bee starts to spin with angular acceleration ε and rushed directly up through the air. Its forward speed is always proportional to the angular velocity of the bee. When the motor flames out after time t, the firework charge explodes. What is the distance covered by the tip of the bee's wing?



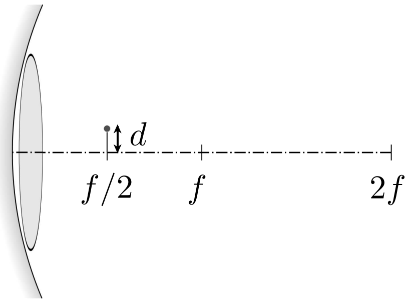
Obr. 10

25. Mike (70 kg) and Claire (56 kg) are climbing an ice mountain from opposite sides. Due to negligible friction they are connected together over the hill by a strong rope. Determine the angle α if Mike and Kate are at rest. Calculate α in degrees to the accuracy of one decimal place.
This problem has no closed-form solutions.



Obr. 11

26. An optical system consists of a thin biconvex lens of focal length f and a concave mirror of focal length $2f$. The distance between the lens and the mirror is negligible. There is a pin of length d at the distance of $f/2$. If we look at the lens we will see an image of the pin. How big is the image of the pin in the mirror? Is it upright or inverted?



Obr. 12

27. Martin played with a very massive and electrically conductive homogeneous spherical shell of radius R and mass M. When charged with total charge Q, the sphere doesn't have a tendency to contract or expand. What is the value of Q?

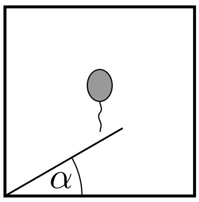
28. An airplane stands still at the beginning of a runway. When the pilot starts the engines, the airplane begins to accelerate with acceleration of $a = 2\,\mathrm{m\,s^{-2}}$. The airplane has to achieve a speed of $v = 80\,\mathrm{m\,s^{-1}}$ with respect to the air to take off. For a headwind takeoff, the airplane needs a runway $L = 1200$ metres long. How long should the runway be for a tailwind takeoff?

29. George made a slingshot. He used an elastic band with zero rest length, stiffness k and a Y-shape branch with span d. What is the stiffness (ratio of the force applied and the resulting displacement) of the slingshoti, if we pull the centre of the elastic band?

30. We take a piece of a steel rope of length L. The tensile strength of the steel rope has such value that a freely hanging piece of length L would snap. Now we take another part of the same rope and we spin it horizontally with a large angular velocity ω while holding one of its ends. What is the maximal length of the rope in this case if we do not want it to snap?

You may neglect gravitational force.

31. Johnny is sitting in the corner of a square shaped room with a side of 10 m. All four inner walls of the room are covered with mirrors and a small balloon is placed in the centre. Johnny would like to illuminate the balloon with a laser pointer, but he is scared the balloon might pop.
Fortunately, the air inside the room is full of dust and the mirrors are dirty. The intensity of laser light drops 1 % for each metre travelled and 10 % after each reflection from a mirror. The intensity of Johnny's laser is exactly twice as much as it is needed to pop the balloon. At what angle α should Johnny shoot the laser pointer in order to maximise the intensity of the incident laser light, if he does not want to pop the balloon?



Obr. 13

Laser has to be directed inside the room ($0° < \alpha < 90°$).

32. The new ecological vehicle "Bathmobile" consists of a bathtub filled with water of surface area A which can move almost frictionlessly on rails. In the rear part it has a nozzle that squirts water and thus pushes the Bathmobile forward. The area of the nozzle's cross section is S. The ability to refill the water in the bathtub during rain is certainly a big competitive advantage.
If the intensity of the rain is w (measured e.g. in $\mathrm{mm\,h^{-1}}$), the velocity of the vehicle and the height of the water inside the bathtub are soon stabilized. Your task is to find this stabilized velocity. You may assume that rain falls directly downwards.

33. Jacob found out that when current I flows through his wire of the radius r, the wire is heated to the temperature T. One day, Matthew gave him a present: a second wire (same material, same length) whose radius was $2r$. How many times should Jacob increase the current I through his new wire if he wants to retain the same temperature T as before? Assume that the heat is emitted only from the cylindrical shell of the wire.

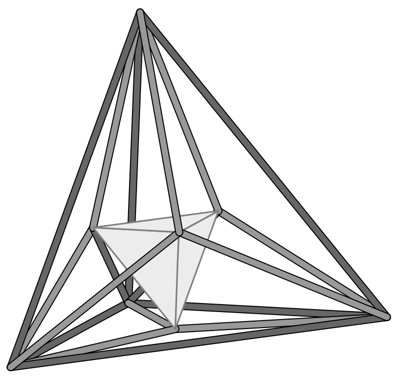
34. Let's assume the planets orbit the Sun in a single plane and on circular orbits. The radius of Venus's orbit is $k = 1.4$ times smaller than that of Earth's. What is the duration of the transit of Venus? Consider Earth and Venus to be points and assume that the angular diameter of the Sun is sufficiently small.

Submit numerical result.

35. The exoplanet *Cimermanos* has radius R_c, temperature T_c and orbits around its parent star at a distance of D_c. Young astronomer Jacob has discovered another planet orbiting the same star. The newfound planet has orbital period τ_n eight times larger and radius R_n three times larger than *Cimermanos*. What is the temperature T_n of the newfound planet, if they and their parent star are perfect black bodies?

36. Martin flies with Supersonic Airlines to the east exactly above the equator at the altitude of 18,000 m. Nina is watching him from the earth's surface and she observes that the airplane is moving at $600\,\mathrm{m\,s^{-1}}$. When Martin stands on scales in the airplane he sees that he is really heavy – exactly 90 kg. What is the real mass m of Martin if the scales show the exact mass of an object in a gravitational field with acceleration of $g = 9.81\,\mathrm{m\,s^{-2}}$?

37. Tom was watching football on TV. As the opponent of his favourite team hadn't put up any resistance, he decided to calculate the resistance between two vertices of the tetrahedron shaped structure depicted below. The outer construction is made of resistance wire which connects all vertices of the tetrahedron and centres of each side with their respective vertices. The resistance of each wire is R. Furthermore, the centres of all sides are connected to a smaller tetrahedron, which is made of perfectly conductive metal sheet.

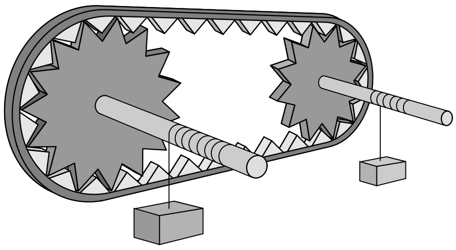


Obr. 14

38. Tom decided to climb Mont Blanc. His trip began in the village Chamonix at an altitude of 1000 m above the sea level. There he bought a croissant in a pack whose volume was 280 ml. During the climb the pack reached a maximal volume of 300 ml. At what altitude did the pack burst?
The pack is able to resist a pressure difference of 10 kPa, the volume of the croissant itself is 180 ml. Assume that the atmosphere around Mont Blanc is isothermal and the pressure is decreasing exponentially according to the formula $p = p_0 \cdot e^{-ch/p_0}$, where p_0 stands for the atmospheric pressure at sea level and C equals $10\,\mathrm{Pa\,m^{-1}}$.

39. Consider two heavy cogwheels of masses M_1, M_2 and radii R_1, R_2 connected with a chain. Rods with a wound rope are attached to each wheel in such way that the ropes hang as

shown in the picture below. The radii of the rods are ρ_1, ρ_2 and masses μ_1, μ_2 are attached to the ends of the ropes. At the beginning, the system is kept at rest by holding the masses. What is the acceleration of mass μ_1 at the moment when the masses are released? You may assume that the moment of inertia of the cogwheel is the same as the moment of inertia of a disc of equal radius.



Obr. 15

40. The tensile strength of a steel rope has such value that a freely hanging part of length L snaps. Jacob found a piece of length L and made it into a loop. Then he placed it horizontally and started spinning it around the vertical axis passing through the loop's centre. At what angular velocity does the rope snap?

41. Martin went camping this summer. One night he noticed a very fast and bright meteor. Later he learnt that most meteors are grains of dust fallen off passing comets, and that this particular meteor must have originated from a certain comet, whose orbit around the Sun had a parabolic shape. Can you tell what the maximal velocity of the meteor could have been when it entered the Earth's atmosphere?
Assume that the Earth's orbit is circular and the thickness of the atmosphere is negligible.

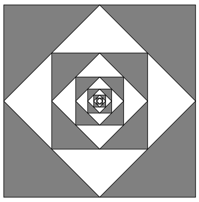
42. Jacob and Matthew were observing distant objects in the Universe. Suddenly, they noticed a fast moving unknown object heading toward the Earth, so they decided to measure its velocity. The measured apparent velocity was $3c$, even though they knew that the real velocity must be lower than c. Find the apparent velocity of the object when moving away once is passes the Earth.

43. Caroline was playing with a very interesting physical system. She took a thermally isolated box with perfectly reflective inner surface and put a radiator of temperature $T = 3000\,\mathrm{K}$, area S and practically infinite heat capacity inside. The box also contains a complicated optical device that focuses all the power emitted by the radiator onto a small black slat of area $S/10$. What will the temperature of the slat T_d be once the system reaches thermal equilibrium?

44. In a galaxy far far away, there lies a star of mass M and radiating power P. A planet of density ρ and radius R is orbiting the star. Its orbital period is T. What is the radius r of the planet's orbit?

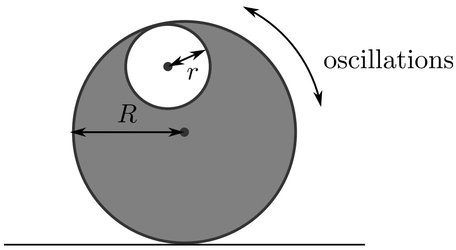
45. Mike brought his new finding – a so called "Mike's square" – to the office. The edge length of the square is a and its mass is M. The process of making "Mike's square" is as follows:

Take a full square with edge length a. Then cut off a smaller square whose vertices are lying in the centres of the original square's edges. Then add another smaller square of smaller length whose vertices are lying in the centres of the empty square's edges. Repeat forever. You should obtain the shape depicted below. Mike's friend, Matthew, was so excited about this shape that he immediately calculated its moment of inertia around its axis of symmetry perpendicular to the plane of the square. What was the result?



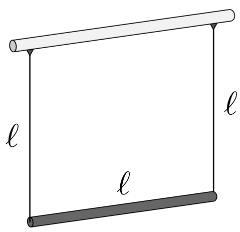
Obr. 16: Mike's square

46. Patrick's wheel is broken. The shape remained the same, but a small hole of radius $r = R/3$ appeared near the rim. Determine the period of small oscillations of the wheel when we put it on the ground and let it roll from side to side around the equilibrium. The radius and the mass of the broken wheel are R and M respectively.



Obr. 17

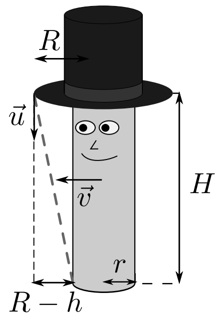
47. A rectangular frame is composed of four thin rods of length l. The lower one is conductive, the side ones are non-conducting and their masses are negligible. The upper one is horizontal and fixed, so that the entire frame is able to rotate about the upper rod. The frame is placed in a horizontal magnetic field $\vec{\mathbf{B}}$, which is perpendicular to the rods. Suddenly, little Jimmy appeared and pulled the frame to horizontal position. Then it started to oscillate. Determine the highest voltage U_{max}, which can be induced in the lower rod during the motion.



Obr. 18

Solutions

$\boxed{1.}$ A rain drop falling next to the hat overcomes height H in time $t = \dfrac{H}{u}$. If the drop is not to fall on Jimmy, he can't move more than $R - r$ in this time. Therefore $R - r = vt$. Plugging in the expression for time, we get Jimmy's maximal speed is $v = \dfrac{R - r}{H} u$.



Obr. 19

$\boxed{2.}$ Water flows from a tap only if it is pushed by greater pressure than the outer pressure is. Thus the pressure in the top part of water pipe must be higher than 1 atmosphere. That means that the pressure can drop by 19 atmospheres and the water keeps flowing from the tap. The reason for the pressure decrease in the water pipe is hydrostatic „counter“pressure caused by a water column above ground floor. Therefore, the maximal height of the building is

$$h = \frac{19\,\mathrm{atm}}{\rho g} = \frac{1{,}925{,}175\,\mathrm{Pa}}{1000\,\mathrm{kg\,m^{-3}}9.81\,\mathrm{m\,s^{-2}}} \approx 196\,\mathrm{m}\,.$$

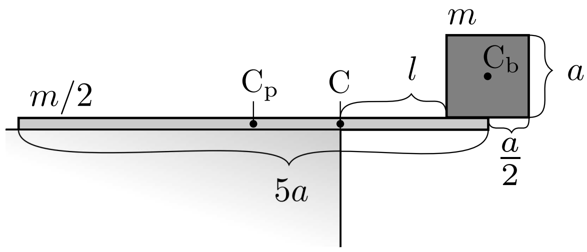
$\boxed{3.}$ With every fold the thickness doubles. At the beginning it is $d = 10^{-4}\,\mathrm{m}$ and after n folds it is $d \cdot 2^n$. We have to reach thickness of at least $1\,\mathrm{AU} = 150 \times 10^9\,\mathrm{m}$, so we need find N such that $1\,\mathrm{AU} = d \cdot 2^n$. This is easy, we only have to take the logarithms [1]

$$\log_2\left(\frac{1\,\mathrm{AU}}{d}\right) = \log_2\left(2^N\right) = N \qquad \longrightarrow \qquad N = \log_2\left(\frac{1\,\mathrm{AU}}{d}\right) \doteq 50.4$$

Rounding the result up we see that we need to fold the the paper 51-times in order for the thickness to reach the distance between the Sun and the Earth.

$\boxed{4.}$ The cube won't fall off the plank if its centre of gravity (CoG) is above it – the furthest such position is at the end of the plank. The plank and the cube won't fall off the cliff if their combined CoG is on the ground. This is, in extreme, satisfied when the CoG lies directly above the edge of the cliff. To solve the problem, we must find the position of the CoG.

[1] We can choose a logarithm of any basei, but two is obviously the most practical.



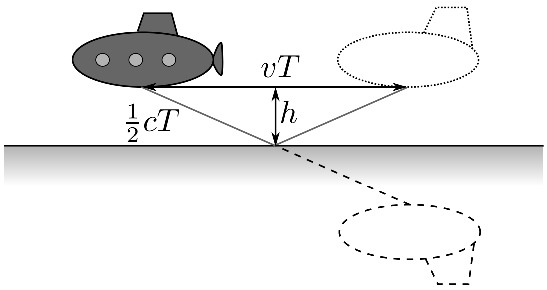
Obr. 20

We will measure the distances from the edge of the plank. The CoG of the plank is at the distance $T_p = \frac{5}{2}a$ and the CoG of cube at the distance $T_c = 5a$. The position of the common CoG is the weighted average of positions of CoGs of the plank and the cube, $\dfrac{\frac{m}{2}\cdot\frac{5}{2}a + m\cdot 5a}{\frac{m}{2}+m} = \frac{25}{6}a$.

Hence, the largest distance is $l = 5a - \frac{25}{6}a - \frac{1}{2}a = \frac{1}{3}a$.

$\boxed{5.}$ Let's consider short time period τ. During this period, water of volume $V = Q\tau$ and mass $m = \varrho Q\tau$ flows through the heater. The energy $E = mc\,(T_{hot} - T_{cold})$ is needed to heat this amount of water from temperature T_{cold} to T_{hot}. Therefore, the power of the heater is $P = E/\tau = \varrho c Q\,(T_{hot} - T_{cold})$. Using numerical values from the problem formulation this yields $2.09\,\mathrm{kW}$.

$\boxed{6.}$ If the velocities of both sound and the submarine are constant, there is only one angle α under which the ultrasound could have left the submarine so that its reflection is detected after T seconds (see figure).



Obr. 21

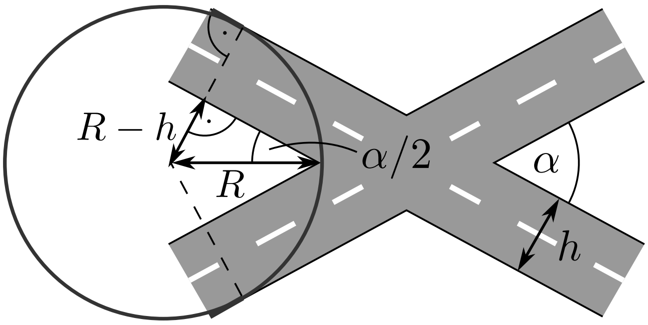
The following holds for this angle: $\cos\alpha = \frac{vT}{cT} = \frac{v}{c}$. Looking at the right-angled triangle containing this angle the depth of the horizontal bottom follows from Pythagoras' theorem:

$$h = \frac{cT}{2}\sqrt{1 - \frac{v^2}{c^2}}$$

.

$\boxed{7.}$ Consider first the situation, when the ice cube floats at the surface of the drink. The glass-drink-ice system forms a single object of mass $m_g + m_d + m_i$. The correspnding weight is measured and displayed by the scales. To submerge the ice cube, Kate must exert a force F, which is the difference of buoyancy and the force of gravity acting on the ice cube, $F = F_b - F_G$. The difference between this and the first situation lies in the presence of the force F which will affect the weight reading on the scales. The new weight will be $m_g + m_d + m_i + \dfrac{F}{g}$, so the difference is $\Delta m = \dfrac{F_b - F_G}{g} = a^3\left(\varrho_d - \varrho_i\right)$.

$\boxed{8.}$ The tractor follows an arc. As the tractor may not leave the road, the outer edges of the road must be tangents to the trajectory of the tractor. Therefore we need to find the largest circle, which touches the outer edges of the road while the arc defined by the tangent points lies entirely on the road. It immediately follows that the point of intersection of the inner edges of the road must lie on the circle we are looking for. The only thing left is to determine its radius.

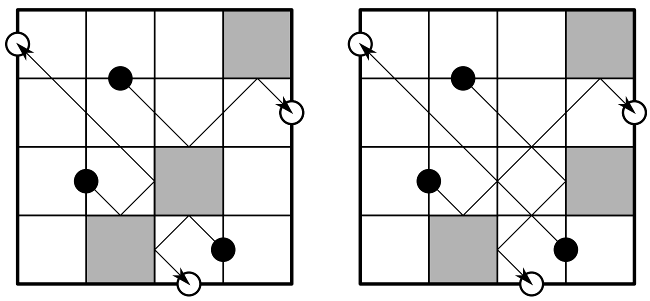


Obr. 22

Using the diagram and simple trigonometry, it follows that $\sin\dfrac{\alpha}{2} = \dfrac{R-h}{R}$, so

$$R = \frac{h}{1 - \sin\dfrac{\alpha}{2}}\,.$$

$\boxed{9.}$ The problem has two solutions:



Obr. 23: Correct positions of mirrors

$\boxed{10.}$ After thinking a little bit about the way balloons fly we should realise that, strictly speaking, they do not. They float at a certain altitude and drift freely with the wind. Therefore we can neglect not only the deformation of the balloon's envelope, but everything else as well, except two parameters – the circumference of the Earth and the (constant) wind speed.

The circumference of Earth is roughly $2\pi \cdot 6378\,\mathrm{km} \doteq 40{,}074\,\mathrm{km}$, and after dividing it by the speed of the air we obtain a flight time of 1002 hours. Rounded to the next greater integer it's 42 days.

After the take-off the balloon accelerates for a certain, non-zero time, though this can be safely ignored as this acceleration takes a time in the order of seconds. The exact calculation is left as an exercise for the reader :-)

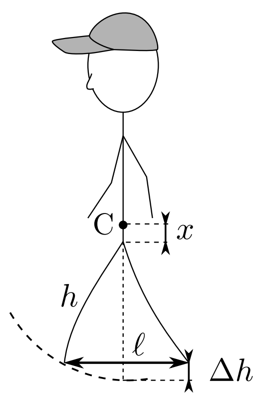
$\boxed{11.}$ Not much can happen here. Either the blocks move together or they slide with respect to each other. The solution is determined by the force F and the force of friction F_f acting between the blocks. The force of friction is a reaction – it opposes the action and its maximal magnitude is

$$F_t = mgf = 3.14\,\mathrm{N}\,,$$

which is more than the action $F = 2\,\mathrm{N}$ acting on the bottom block! Thus the force of friction is able to compensate for it and therefore the blocks move together. The acceleration is

$$a = \frac{F}{M+m} = 0.4\,\mathrm{m\,s^{-2}}$$

$\boxed{12.}$ Let's calculate the energy needed to be expended in one step.



Obr. 24

First, we stand straight so the centre of mass is in the height $x + h$ where x is the distance between the centre of mass and loins. Then we move forward and straddle legs to span l. The centre of mass descends to $x + \sqrt{h^2 - \frac{l^2}{4}}$, thus the change of height of the centre of mass is $\Delta h = h - \sqrt{h^2 - \frac{l^2}{4}}$. Up to this moment, we haven't done any work – all the work has been done by gravitational force. Then we start to straighten again so the centre of mass rises to h and overcomes the height difference Δh. This is when we expend energy in order to overcome the gravitational force. The expended energy is thus $E_0 = mg\Delta h$. If we want to cover the distance s we need $n = \frac{s}{l}$ steps. The expended energy is then $E = nE_0 = \frac{s}{l}\left(1 - \sqrt{1 - \frac{l^2}{4h^2}}\right) mgh.$

$\boxed{13.}$ George's voltmeter shows a voltage decrease on resistors with resistances R and $2R$. That's why we firstly calculate currents flowing through them.

The electric current flowing through resistance R is the overall current in the circuit. What is more, the resistors are connected only in series and parallel, thus it is not difficult to calculate total resistance of the circuit

$$R_0 = R + \frac{(2R + 4R)(3R + 5R)}{2R + 4R + 3R + 5R} = \frac{31}{7}R$$

We get magnitude of current I_0 as $I_0 = \frac{U}{R_0} = \frac{7}{31}\frac{U}{R}$.

The current flowing through the resistor $2R$ can be expressed from Kirchhoff's 2nd law for a closed loop. Alternatively, we know that the current is divided in reciprocal of the ratio of branches' resistances, therefore

$$I' = \frac{3R + 5R}{2R + 4R + 3R + 5R} I_0 = \frac{4}{31}\frac{U}{R}$$

Finally, we can calculate voltage measured by voltmeter as a sum of particular voltage decreases on individual resistors.

$$U_{Volt} = RI_0 + 2RI' = \frac{15}{31}U$$

$\boxed{14.}$ Blergh! A conical frustum... finding the centre of gravity will be rather disgusting here. Unless we find out we do not need it at all. All we need is to realise that torque $\vec{\tau}$ is defined as cross product of the position vector of the point of action $\vec{r}$ and the force $\vec{F}$. $\vec{\tau} = \vec{r}\times\vec{F}$. Therefore, in order to calculate the magnitude of the torque, we only need to know the magnitude of the perpendicular component of the force. Thus we need to balance the torques of all forces related to the bottom edge of the glass and then express the volume. If the mass of the frog is M, the mass of the glass m, upper radius of glass R and bottom radius r, then

$$mgr + V\rho gr = Mg(R - r) \implies V = \frac{M}{\rho}\frac{(R - r)}{r} - \frac{m}{\rho} = \frac{50}{3}\mathrm{ml} \doteq 16.67\,\mathrm{ml}.$$

$\boxed{15.}$ In one cycle the generator burns fuel of mass m_0 and heating value H, which releases energy $E_0 = Hm_0$. Efficiency of the generator is η. This means that we effectively use energy $E = \eta E_0$ and the rest is absorbed as heat $Q = E_0 - E = Hm_0\,(1 - \eta)$ The generator is then cooled by water from the tank with mass m which absorbs all of the heat Q and transfers it into the the tank. We could of course calculate the temperature of the water of mass m and then the temperature after mixing in the tank, but that is quite unnecessary. The entire system is completely insulated and no heat is exchanged with the surroundings, it is therefore contained by the tank. After n cycles heat nQ is released and absorbed by water of mass M and initial temperature t_0. From this we see that $nQ = Mc\,(t - t_0)$ thus $nQ = Mc\,(t - t_0)$

16. The force of surface tension $F_p = 2\sigma d$ acts on the shorter edge of the soap film (The factor of two is due to two interfaces between the soap film and the air.) After the soap film bursts along its shorter edge, this force is no longer compensated by the frame and the film of mass $m = ldh\rho$ starts to accelerate. The acceleration of the soap film is $a = F/m = \frac{2\sigma}{lh\rho} \approx 600\,\mathrm{m\,s^{-1}}$, so even if it were oriented in the direction of the gravitational force, the result wouldn't change by much.

Thus the film moves with a constant acceleration and its centre of mass shifts from the centre of the frame to its edge by a distance $l/2$. Analogically to the expression for the time of free fall,

$$t = \sqrt{\frac{2h}{g}} \implies t = \sqrt{\frac{l^2 h\rho}{2\sigma}} = 13\,\mathrm{ms}$$

If we considered the motion of the free edge of the film, which moves along the longer edge of the frame by a distance l, we would obtain a result of 18 ms. This is the upper bound on the time of shrinking. In contrast, the time 13 ms is the lower bound. The difference is the consequence of our choice of the point, whose motion we consider - the centre of mass or the edge of the film. [2]

17. If we consider horizontal direction, ship is symmetrical and therefore its centre of mass has to be placed in its axis of symmetry. Determination of vertical position of centre of mass is a bit more tricky. Let's recall one of the possible definitions of the centre of mass: When we support the ship in its centre of mass, net torque acting on ship is zero. Therefore we use the balance of torques to find the vertical position of centre of mass.

We split the ship into four parts. First part is upper triangular „roof“. Its centre of mass has vertical coordinate $y_1 = \frac{4}{3}$, with surface area $S_1 = \frac{\sqrt{2}^2}{2} = 1$. Furthermore, we have two triangles at the edges with surface area $S_2 = \frac{1}{2}$ with vertical coordinates of mass centres $y_2 = \frac{2}{3}$.[3] We are left with a rectangle with surface area $S_3 = 3$ and centre of mass with vertical coordinate $y_3 = \frac{1}{2}$.

[2] If we wanted to obtain a more accurate result, we would also have to take into account that the mass of the moving soap film is changing.

[3] Median pointing to the hypotenuse has length of $\frac{1}{sqrt2}$ and centre of mass is situated in one third of this distance from hypotenuse. After calculation of perpendicular distance from a leg we get horizontal position $\frac{1}{3}$.

Now we have all the necessary information and if we assume surface density of paper to be constant across its entire surface, we obtain the following result for the vertical position of centre of mass.

$$y = \frac{S_1 y_1 + S_2 y_2 + S_3 y_3}{S_1 + S_2 + S_3} = \frac{7}{10}$$

Thus the coordinates of the centre of mass are $[2.5; 0.7]$.

$\boxed{18.}$ We can split the problem into two parts. In the first the bullet's flight is uniformly decelerated, with initial velocity v upwards, which takes time t_1. In the second, after the impact, the sound wave propagates with constant velocity c towards us, which takes time $t - t_1$, where $t = 5\,\mathrm{s}$. In both cases, the distance travelled is h, precisely the height we are looking for

$$h = vt_1 - \frac{1}{2}gt_1^2$$
$$h = c(t - t_1)$$

From which we get an ugly quadratic equation and its solutions for the height of the cave.

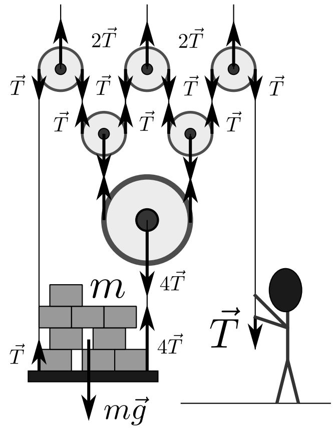
$$h = \frac{\frac{gt}{c} - 1 - \frac{v}{c} \pm \sqrt{\left(1 + \frac{v}{c} - \frac{gt}{c}\right)^2 + 4\frac{g}{2c^2}\left(vt - \frac{1}{2}gt^2\right)}}{\frac{g}{c^2}}$$

At this point, we need to stop and think about which solution is the one we need. Only the one with positive height makes sense, therefore the solution with a plus sign. Simplifying the expression, at last we get the height of the cave.

$$h = c\left(t - \frac{c+v}{g} + \sqrt{\left(\frac{c+v}{g}\right)^2 - \frac{2ct}{g}}\right) \approx 591.20\,\mathrm{m}$$

$\boxed{19.}$ As the problem is stated, we can neglect the mass of ropes and pulleys and can consider them to be ideal. However, this means that the sum of forces and torques acting on every pulley is zero and the ropes transfer force, which strains the rope along all of its length.

Now that we know this, imagine that we pull the rope with force T. Because the rope transfers force, the bricks are being pulled by this rope with force T. The pulleys in upper row are pulled by forces T downwards from both sides and their mounts compensate this with force $2T$. Thus the equilibrium of forces is achieved on these three pulleys. Forces T pull the two pulleys in the centre upwards, and therefore the rope hanging from them has to be strained with force $2T$ on both sides. From this consideration, we see that the rope hanging from the last pulley acts on the bricks with force $4T$.



Obr. 25: Forces

A total force $5T$ therefore acts on the bricks, which must be larger than mg. So if we want to lift the bricks we need to apply a force at least $\frac{mg}{5}$.

20. After each dilution, the number of molecules of active substance drops to one hundredth (on average). If we denote the number of molecules by N we have to know what is the minimal number of dilutions n (integer) to make the number of molecules of active substance smaller than one (on average). $N\left(\frac{1}{100}\right)^n < 1$. We use the logarithm to do so.

$$n = \lceil \log_{100} N \rceil$$

4

Now we only have to find out how many molecules are in volume $V = 5\,\mathrm{ml}$ of water. From Avogadro constant ($N_A = 6.022 \times 10^{23}\,\mathrm{mol^{-1}}$) and molar mass of water ($M_m = 18\,\mathrm{g\,mol^{-1}}$) we can calculate it to be

$$N = \frac{V\rho}{M_m} N_A$$

Thus on average we need at least

$$n = \left\lceil \log_{100}\left(\frac{V\rho N_A}{M_m}\right)\right\rceil = \left\lceil \log_{100}\left(\frac{5\times 10^{-6}\,\mathrm{m^3}\cdot 1\times 10^{3}\,\mathrm{kg\,m^{-3}}\cdot 6.022\times 10^{23}\,\mathrm{mol^{-1}}}{18\times 10^{-3}\,\mathrm{kg\,mol^{-1}}}\right)\right\rceil = 12$$

dilutions.

21. We are asked to find the angular velocity of the roll – we can expect to find the law of conservation of energy useful. In our situation the potential energies of the cat and toilet paper are transformed into kinetic energies of the cat and paper; no energy is transferred to the roll as its mass is negligible.

In terms of equations:

[4] Where $\lceil x \rceil$ denotes the the whole upper part of x

$$-\Delta E_p = \Delta E_k$$
$$MgL + mg\frac{L}{2} = \frac{1}{2}Mv^2 + \frac{1}{2}mv^2$$

Using these two we can easily find the velocity of the cat, paper and the tangential velocity of the roll at the moment when all the paper is unwound, $v = \sqrt{\frac{2M+m}{M+m}Lg}$. Thus the angular velocity of the roll at this moment is:

$$\omega = \frac{v}{r} = \frac{1}{r}\sqrt{\frac{2M+m}{M+m}Lg}$$

$\boxed{22.}$ The jet of water narrows due to continuity equation – the flow Q is conserved. That is

$$Q = \text{const.} \qquad \longrightarrow \qquad Q = \pi R^2 v_0 = \pi r^2 v_r\,,$$

where $v_0 = \frac{Q}{\pi R^2} \doteq 16\,\mathrm{cm\,s^{-1}}$ is the velocity of water flowing from the tap, v_r is the velocity of water at the point where the radius of the jet is r. We need to find the velocity of water at the basin floor. The energy of any small volume of water which reaches the basin floor is conserved, so:

$$\frac{1}{2}mv_0^2 + mgH = \frac{1}{2}mv_r^2 \qquad \longrightarrow \qquad v_r = \sqrt{v_0^2 + 2Hg}$$

We substitute for v_r in the continuity equation and obtain:

$$Q = \pi r^2\sqrt{v_0^2 + 2Hg} \qquad \longrightarrow \qquad r = \sqrt{\frac{Q}{\pi\sqrt{v_0^2 + 2Hg}}} = \doteq 0.50\,\mathrm{cm}$$

$\boxed{23.}$ Let's think about the situation, one car at a time. The first car will start moving instantaneously. The second car will start moving after the first rope gets fully stretched, which means after time l/v_1. The speed of the two moving cars v_2 can be obtained from the law of conservation of momentum. [5]

$$mv_1 = (m+m)v_2 \qquad \longrightarrow \qquad v_2 = \frac{v_1}{2}$$

The speed was halved. Let's consider another car. The third car starts to move when the second rope is stretched, i.e. after time $l/v_1 + l/v_2 = 3l/v_1$. The speed of three moving cars can be once again obtained from the law of conservation of momentum.

$$mv_1 = (m+m+m)v_3 \qquad \longrightarrow \qquad v_3 = \frac{v_1}{3}$$

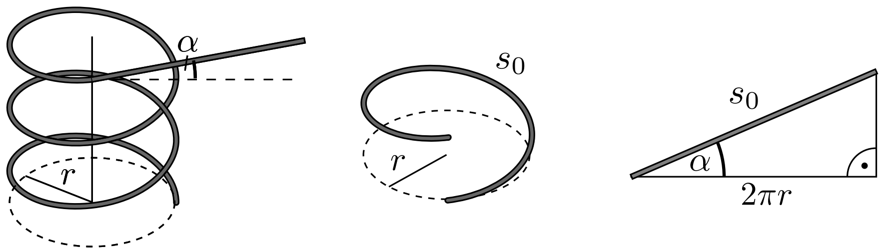
[5] We can't use the law of conservation of energy as the process of setting a car into motion is essentially an inelastic collision.

It's straightforward to extend this to any number of moving cars. A train of n cars will move with a speed v_1/n after the $(n-1)$-th rope is stretched. The entire Michael's train will move after time

$$T = \frac{l}{v_1} + \frac{l}{v_2} + \cdots + \frac{l}{v_{N-1}} = \frac{l}{v_1}\Big(1 + 2 + \cdots + (N-1)\Big) = \frac{l}{v_1}\sum_{i=1}^{N-1} i = \frac{l}{v_1}\frac{N(N-1)}{2}$$

$\boxed{24.}$ The angular acceleration of the "bee" is ε. In time t it rotates by an angle $\varphi = \frac{1}{2}\varepsilon t^2$, which is $n = \frac{\varphi}{2\pi} = \frac{\varepsilon t^2}{4\pi}$ revolutions.

As stated in the problem, the forward speed of the "bee" is proportional to its angular velocity. After a little thought we find out that the tip of a wing traces a helical path of constant pitch. The helix angle is determined by the inclination of the wings, therefore it's equal to α. Let's denote the length of one turn by s_0. The projection of a turn into a plane (a circle) has a circumference $2\pi r$. Now, consider a straightened turn – a segment of length s_0 at angle α to the horizontal plane, and its projection. We obtain the right triangle given in fig. (26).



Obr. 26

It is evident that $\cos\alpha = \frac{2\pi r}{s_0}$, and therefore $s_0 = \frac{2\pi r}{\cos\alpha}$. The distance covered by the tip of the wing is obtained by multiplying the length of a turn by the number of turns. We have $s = ns_0 = \frac{\varepsilon r t^2}{2\cos\alpha}$.

$\boxed{25.}$ We denote Mike's mass M and Claire's mass m. If we decompose components of gravitational force acting on Claire and Mike, we find out that components of gravitational forces acting in the direction of the slope have to be balanced.

$$mg\cos(\alpha) = Mg\cos(2.5\alpha)$$

After substituting masses of both climbers, we are left with following equation:

$$cos(\alpha) = 1.25\cos(2.5\alpha)$$

However, there is a problem! If we attempt to solve this equation analytically by using goniometric formulas, we obtain polynomial of a high degree. There is no general solution for such polynomial, therefore analytical solution would be rather cumbersome (a.k.a. time-consuming).

Because we are looking for a solution of equation for specific values of masses (and, well, we seek only numerical values), we use the calculator. We just want to find the solution of the problem as quickly as possible. This is common case in physics: We obtain equations which do not have a general solution, but they can be solved only for some specific values.

Probably the most suitable method is *interval bisection method*. Don't worry, it is nothing you should be afraid of! First, from our equation, we obtain a function $f(\alpha) = \cos(\alpha) - 1.25\cos(2.5\alpha)$. Now, we will observe how the sign of this function changes in given interval. If the function changes sign in the interval, there is definitely a root in given interval.[6] The main idea of this method lies in the reasonable reduction of this interval. For instance, required accuracy might be such that initial point and endpoint of the interval are equal in the second decimal place in degree measure.

1. To start, let's guess the interval $< x, y >$ which the solution belongs. It is simple for this particular problem. If solution exists, it must lie somewhere between 0 and $\frac{1}{2.5}\left(\frac{\pi}{2}\right)$.[7]

2. Now we look at the value of function $f(\alpha)$ in point $\alpha = \frac{x+y}{2}$. The sign of the function in this point tells us which interval we should examine in the next step. Sign of the function should be altered in this interval, thus we change either x or y to the new midpoint of interval.

3. Outlined procedure is repeated until this interval becomes small enough. In this particular case, it is until endpoints of interval differ only in the first decimal place.

It does not challenge our thinking as much as our patience! After thirteen repetitions, we obtain the desired answer: 15.9°. Computers use approaches that require lower number of steps, but the work on such calculations would be overwhelming and slow! And being slow is really a less-than-optimal strategy for Náboj.

Another possible way how to obtain the result is use of Taylor expansion[8], so $\cos(\alpha)$ and $\cos(2.5\alpha)$ are approximated by polynomials, for convenience in neighbourhood of zero. We have cosines on both sides of equation, we obtain only even powers in Taylor expansion. However, if

[6] To be exact, this statement is valid only for continuous functions – such functions whose graph can be drawn „at once.“ (speaking in popular manner :))

[7] Angle α must not be greater than $\frac{1}{2.5}\left(\frac{\pi}{2}\right)$ (36°), because no equilibrium could be reached otherwise. In such case, Claire would pull poor Mike to the valley under the mountain. Think about it (now we mean physics rather than details of Mike's fate)!

[8] Taylor expansion for function $f(x)$ in neighbourhood of point x_0 is $f(x) = f(x_0) + \frac{f(x_0)'}{1!}(x - x_0) + \frac{f(x_0)''}{2!}(x - x_0)^2 + \ldots$, where ' designates differentiation of function in given point.

we had an equation which contains both sines and cosines, we could not use such trick because we would obtain a cubic equation again.

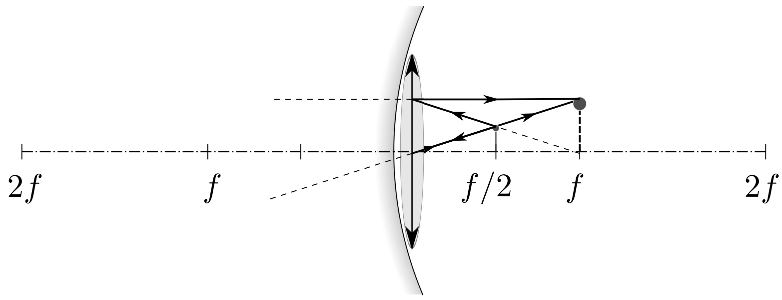
$$\cos(\alpha) \approx 1 - \frac{x^2}{2} + \frac{x^4}{24} + \dots$$
$$\cos(2.5\alpha) \approx 1 - 3.125x^2 + 1.6276x^4 + \dots$$

After substitution, we obtain a biquadratic equation,

$$\cos(\alpha) - 1.25\cos(2.5\alpha) = 0 \implies x^4 - 1.70925x^2 + 0.12545 = 0$$

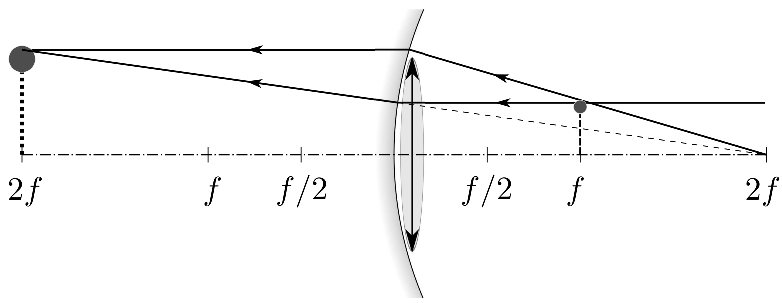
We take the solution $\alpha = 0.277218$ (in radians) lying in our interval. After conversion to degrees, we obtain the value 15.9°. Recalling the required accuracy, we can consider this result to be correct :) [9]

$\boxed{26.}$ The images will be sequentially created by the convergent lens and mirror until we get real image on our side. The rays are marked in the picture. First, we create image of the pin by convergent lens with focal length f. Image of the pin will be then placed into the focus of the convergent lens at the same side. It will get to the two times greater distance as before and image will therefore be two times larger.



Obr. 27: Creation of the image of the pin by convergent lens with focal length f.

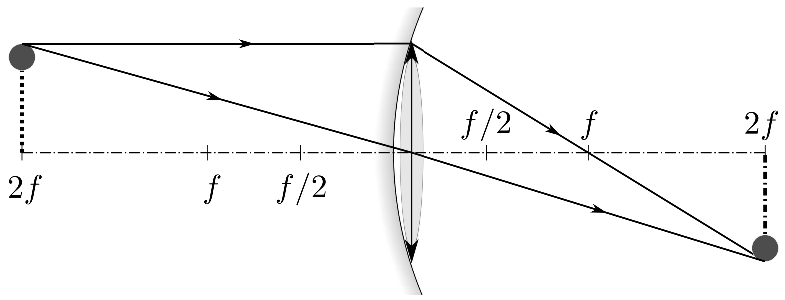
Subsequently, image of the pin created by the convergent lens will be displayed by the convex mirror with focal length $2f$. Our image will then become virtual, it will get to the two times larger distance than it was before and it will be again twice as big as before.



Obr. 28: Creation of the image of the pin displayed by the convergent lens by the concave spherical mirror.

[9] The discrepancy is only visible in the third decimal place :).

Finally, we depict virtual image of the pin by converging lens, again, and picture will become real and neither its distance from the middle of the convergent lens nor its size will change.



Obr. 29: Portrayal of the virtual image of the pin by convergent lens.

Overall, we obtain image of the pin in distance of $2f$ from the convergent lens. The image will be inverted and four times magnified.

27. This problem is not particularly difficult, but requires considerable physical insight.

First we need to understand why the shell tends to contract. Consider a very small element of the shell. As gravitational force is attractive and the mass of the shell is homogeneously distributed, gravitational force of the rest of the shell (all other elements) acts on the element. The resultant force pulls the element inward, thus the shell tends to contract. Analogously, the interaction between shell elements of a massless shell homogeneously charged with a charge Q would cause the shell to expand. We will denote the charge per unit area by $\sigma = \dfrac{Q}{4\pi R^2}$ and the mass unit per unit area by $\lambda = \dfrac{M}{4\pi R^2}$.

What is going to happen if the shell has nonzero mass and is charged? The key facts are that the magnitude of the gravitational and electrostatic force is proportional to the inverse square of the distance and that they point in exactly opposite directions. So we're dealing with two "identical" fields differing in orientation and physical constants, which ensure that the fields have correct dimensions.[10] Now consider two shell elements of area S_1 and S_2. The shell must be charged with a charge Q such that gravitational and electrostatic forces cancel each other out.

$$F_{\text{gravitational}} = F_{\text{electrostatic}} \implies G\frac{\lambda^2 S_1 S_2}{r^2} = \frac{1}{4\pi\varepsilon_0}\frac{\sigma^2 S_1 S_2}{r^2} \implies GM^2 = \frac{1}{4\pi\varepsilon_0}Q^2$$

$$Q = \sqrt{4\pi\varepsilon_0 G}M$$

[10] However, it is necessary to note that this analogy is limited. Mass is always positive and there are no phenomena similar to the polarity of charge. Furthermore, the variety of types of matter is not as rich as in electromagnetism; consider just insulators and conductors, which are responsible for many interesting phenomena of electrostatics :).

28. We are actually looking to an airplane from two different frames of reference. The first is connected with the Earth, the second is from the point of view of the air. They are moving to each other with some, a priori unknown, velocity u.

Let's look at the point of takeoff in both frames of reference. In frame of Earth the airplane accelerates from the rest to velocity $v - u$, in frame of the air the airplane accelerates from velocity u to v. The time is obviously the same in both frames of reference, $t = \frac{v-u}{a}$. We can easily obtain the distance covered by airplane during acceleration in frame of the Earth, which has to be equal L.

$$L = \frac{1}{2}at^2 = \frac{1}{2}\frac{(v-u)^2}{a}$$

From the previous equation we can easily obtain $u = v - \sqrt{2aL}$.

Now, let's look at the takeoff in case of tailwind. The only difference is the change of sign of u, since the tailwind. The acceleration will of course take longer, since the airplane has to accelerate from rest to the velocity $v + u$ in the Earth's reference frame. Therefore, L' is also longer.

$$L' = \frac{1}{2}at'^2 = \frac{1}{2}\frac{(v+u)^2}{a}$$

From the previous equation we obtain $u = \sqrt{2aL'} - v$.

Combining both equations we get

$$L' = L + \frac{2v^2}{a} - 2v\sqrt{\frac{2L}{a}}$$

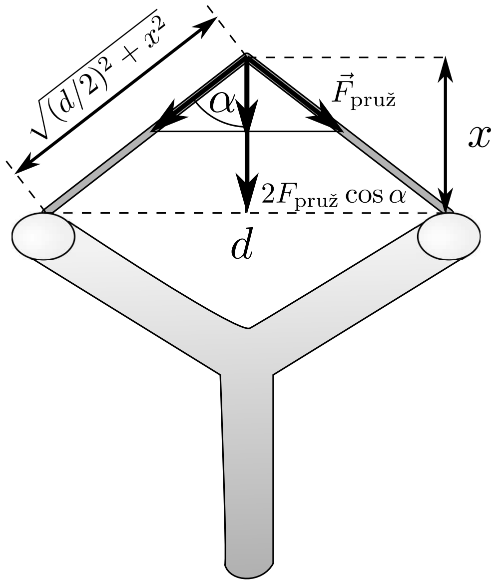
Plugging in numerical values yields the result

$$L' \doteq 2057\,\mathrm{m}$$

.

29. As the problem text suggests, we need to find the dependence of force we need to apply to the centre of the elastic band and the displacement of the band. First, we need to create the slingshot, which stretches the band to length d. Stiffness of the band is k. Notionally splitting the band in half, we see that stiffness of both parts is $2k$.[11] If we displace the centre of the band by x the length of both sides is $\sqrt{\frac{d^2}{4} + x^2}$ (Pythagoras's theorem). The forces from the bands act in their direction so we need their projection. Thanks to this we get $cos\alpha$ term, where α is the angle between the band and the direction of displacement. We find this cosine from the right-angled triangle.

[11] Imagine a long straight rubber band with stiffness k. If we split it in half, stiffness of both parts is $2k$: If we apply a force F and it extends by Δx, this force acts on both parts which extend by $\Delta x/2$. If the force is to stay the same, the parts need to be twice as stiff.



Obr. 30: Projection of the forces

$$F(x) = 2(2k)\sqrt{\frac{d^2}{4} + x^2}\cos\alpha$$
$$\cos\alpha = \frac{x}{\sqrt{\frac{d^2}{4} + x^2}}$$

The ugly square roots cancel each other, and we see that the stiffness of the slingshot (ratio of force F and displacement x) is $4k$.

$\boxed{30.}$ The freely hanging rope snaps when the pressure of gravitational force overcomes breaking strength of the rope. In the language of Maths:

$$\sigma = \frac{mg}{S} = \frac{\rho L g}{S},$$

Where L is maximum length of freely hanging rope, S is its cross-sectional area and ρ is its length density.

Now we need to ask how much rope should we unleash from the disk (let us designate this length as L^*), so that the rope is torn because of the effect of centrifugal force while the rope is rotating with large angular velocity ω. According to the task formulation, ω is large, and therefore we can neglect the effect of gravitation. Again, the pressure of the centrifugal force has to overcome the breaking strength of the rope.

The centrifugal force acting on the whole rope can be calculated as centrifugal force acting on the point with the mass of whole rope, which is placed in the mass centre of the rope. [12]:

[12] In formal approach, we should use the following integral:

$$F_o = \frac{mv^2}{r} = m\omega^2 r = \rho L^* \omega^2 \frac{L^*}{2} = \frac{1}{2}\rho\omega^2 L^{*2}$$

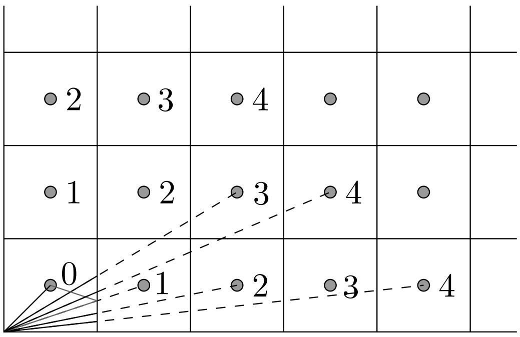
Now we can equate the pressure of the centrifugal force with the breaking strength of the rope:

$$\sigma = \frac{F_o}{S} = \frac{\frac{1}{2}\rho\omega^2 L^{*2}}{S} \qquad \longrightarrow \qquad L^* = \sqrt{\frac{2\sigma S}{\rho\omega^2}}$$

The value of the expression $\frac{\sigma S}{\rho}$ is known from the first equation in this solution, which is $\frac{\sigma S}{\rho} = Lg$. Final result is thus:

$$L^* = \sqrt{\frac{2Lg}{\omega^2}}$$

$\boxed{31.}$ We will use the method of virtual images to solve the problem. Let's extend the room into an infinite grid of virtual rooms and set up a coordinate system. We'll place the origin into the corner where Johnny is sitting. Let the real balloon have coordinates $[5, 5]$. Now we put a (virtual) balloon into the centre of each virtual room and aim our laser pointer at it.



Obr. 31

Denote the number of reflections, equal to the number of walls the laser ray intersects, by k and the distance covered by the ray, equal to the distance to the virtual balloon, by l. Note the reflection symmetry given by the line defined by Johnny and the real balloon.

$$F_o = \int_0^{L^*} \omega^2 r \mathrm{d}m = \int_0^{L^*} \omega^2 \rho r \mathrm{d}r = \frac{1}{2}\rho\omega^2 L^{*2} \,,$$

However, this is not necessary. Since centrifugal force depends linearly on the distance, ($F_o = m\omega^2 r$), we can calculate the area under the graph in similar manner as when we look for the distance travelled during uniform accelerated motion. This area has triangular shape. And area of such triangle in the graph of dependence of centrifugal force on distance from the axis of rotation r is (Think about it!) exactly $\frac{1}{2}\rho\omega^2 L^{*2}$. Notice that we calculate moment of gravitational force in Mechanics in he same manner.

Using the information provided in the text of the problem, we know that the intensity of the ray just before hitting the balloon will be $(0.9)^k \cdot (0.99)^l$ times lower than its initial value. We need k and l, such that $(0.9)^k \cdot (0.99)^l < 0.5$, and that there is no other k and l with a higher intensity.

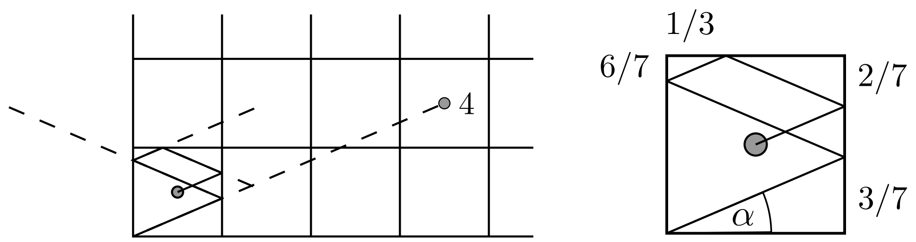
We can hardly just guess the correct solution, so we'll have to try to verify a couple of options. For example, let's illuminate the balloon directly, at an angle ($\alpha = 45°$), so $k = 0$ and $l = \sqrt{5^2+5^2} \doteq 7.07$. The intensity is roughly $(0.9)^0 \cdot (0.99)^{7.07} = 93.1\,\%$ of the initial value. Evidently, this is too high and the balloon would burst. Now we try to illuminate the balloon after one reflection. We need to aim for the virtual balloon at coordinates $[5,15]$. In this case $k = 1$ and $l = \sqrt{15^2+5^2} \doteq 15.8$. Using the expression for the intensity we find out that the ray is still too bright.

Continuing in a similar manner we find out that not even three reflections are sufficient to lower the intensity of the ray, but four are just enough. It appears that the best option is to aim for the image with coordinates $[25,25]$ in the virtual room placed "two right, two above". It's not possible to hit this image as the real balloon is in the path.

Next we try the room "one right, three above" with the balloon image at $[15,35]$. The intensity of the incident ray will be $(0.9)^4 \cdot (0.99)^{\sqrt{1450}} \doteq 44.7\,\%$ of the initial value.

The final option is the room "four above" with the balloon at $[5,45]$. The balloon will be hit by a ray of intensity $(0.9)^4 \cdot (0.99)^{\sqrt{2050}} \doteq 41.6\,\%$ of the initial value. However, this is lower than in the previous case.

More reflections cause a further decrease of intensity both due to reflections themselves and because of increasing length of the path of the ray, so the balloon will not pop. [13] As we were looking for the highest intensity, we found the solution. All what's left is to find the angle. We aim for the virtual balloon at $[15,35]$, so $\alpha = \arctan(15/35) = \arctan(3/7) \doteq 23.2°$ and the intensity is roughly $44.7\,\%$ of the initial value. Using the symmetry of the problem, we know that $\alpha = \arctan(7/3)$ is equally good.



Obr. 32

$\boxed{32.}$ We can obtain the correct solution almost effortlessly; however, we have to think thoroughly about how it works. The bathtub collects falling raindrops, which have zero velocity in horizontal direction and squirts it to the side with certain velocity. It gives momentum to

[13] Careful, this statement is not universally valid. If the drops in the intensity were lower than those given in the problem, a solution with more reflections but a shorter path might be better. It can be easily verified that, for given values, this is not the case.

the water and (because of the law of conservation of momentum) therefore vehicle gains equal momentum in the opposite direction.

What happens in the steady state? 1. Surface level stays constant and 2. Momentum of the vehicle with water stays constant. Therefore:

1. Inflow of the rainwater always has to be equal to the outflow: $wA = v_{outflow}S$.
2. The horizontal component of momentum of the incident rainfalls has to be equal to the horizontal component of the squirted water. Because rainfalls are falling in the vertical direction (zero horizontal component), squirted water has to be in rest with respect to the ground. This happens when water is squirted with velocity equal to the velocity of the vehicle (though in the opposite direction): $v_{vehicle} = v_{outflow}$.

Now, we can immediately see the result: $v_{vehicle} = v_{outflow} = w\dfrac{A}{S}$.

$\boxed{33.}$ In equilibrium state, when temperature doesn't chage, the power of Joule heat ($P_{\text{electric}} = I^2R$) has to be equal to the power radiated as heat. Since in both cases we want to achieve the same temperature, we don't have to think about what type of heat losses is relevant (radiation or heat conduction). The crucial fact is to realise that in both type of heat loss, the power is directly propotional to the surface area of the wire (cylindrical surface).

$$P_{\text{losses}} \sim 2\pi rl \implies P_{\text{straty}} = C2\pi rl$$

We have to also realise, that the resistance of the wire changes with change of the radius as well. From Ohm's law we get

$$R = \rho\frac{l}{S} = \rho\frac{l}{\pi r^2}$$

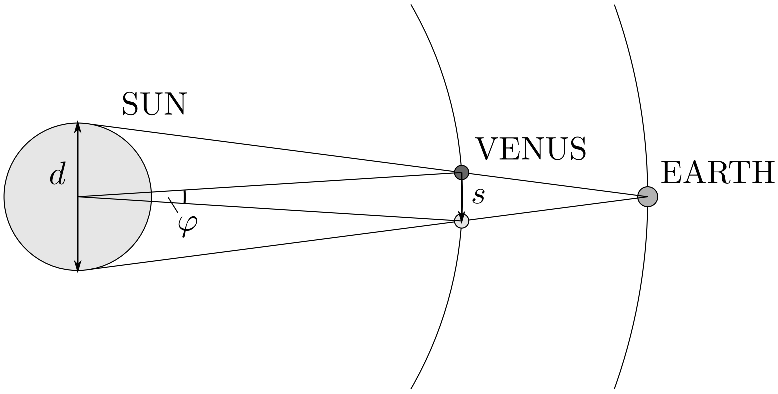
where ρ is measured in Ohm metres and l is the length of the wire.

Comparing the powers

$$P_{\text{electric}} = P_{\text{losses}} \implies I^2\rho\frac{l}{\pi r^2} = C2\pi rl \implies I^2 \sim r^3$$

So we have to increase the current $\sqrt{2^3}$ times in order to make the temperature constant. (Don't forget that C is not dependent on temperature!)

$\boxed{34.}$ Let's assume, for the sake of simplicity, that the Earth and the Venus are point object (as they are small compared to other distances in the problem). Let the distance between the Earth and the Sun be R and the distance between the Venus and the Sun $\dfrac{R}{k}$. We see the Sun under a small view angle. That means the arc that Venus describes during the transit is rather short and we can approximate it with a straight line. From the affinity of triangles we get $\dfrac{2r}{R} = \dfrac{s}{R - \dfrac{R}{k}}$, and then $s = \left(1 - \dfrac{1}{k}\right)2r$.



Obr. 33

Using the assumption of small view angle, there is no significant difference between straight line and a small circular arc. Thus the small view angle is $\varphi = \frac{s}{\frac{R}{k}} = \frac{(k-1)\,2r}{R}$.

The Earth's angular velocity can be determined considering centripetal and gravitational force

$$m\omega_Z^2 R = G\frac{mM}{R^2}$$

. Thus

$$\omega_Z = \sqrt{\frac{GM}{R^3}}$$

. The Earth's angular velocity is $\omega_Z = \frac{2\pi}{T_Z}$. Using the third Kepler's law we calculate transit time of Venus and consequently the Venus angular velocity. Since

$$\frac{T_Z^2}{R^3} = \frac{T_V^2}{\left(\frac{R}{k}\right)^3}$$

, we get

$$T_V = \frac{T_Z}{\sqrt{k^3}} = 2\pi\sqrt{\frac{R^3}{GMk^3}}$$

. Therefore

$$\omega_V = \frac{2\pi}{T_V} = \sqrt{\frac{GMk^3}{R^3}}$$

.

Let's determine the Venus transit time by τ. The Venus has to travel an angle φ, bigger than that of the Earth during this time. Therefore $\omega_V\tau = \omega_Z\tau + \varphi$. Thus

$$\tau = \frac{\varphi}{\omega_V - \omega_Z}$$

and after the substitution of formulas of angle and angular velocity, we get

$$\tau = \frac{2\,(k-1)\,r\sqrt{R}}{\sqrt{GM}\left(\sqrt{k^3}-1\right)}$$

. Numerically,

$$\tau \approx 28{,}440\,\mathrm{s} \approx 7.9\,\mathrm{h} \approx 7\,\mathrm{hour}54\,\mathrm{minute}$$

.

At the end we should also verify the applicability of our assumption. However, it is not difficult to calculate that during the transit the Venus follows an angle $\alpha = \omega_V \tau \approx 0.54^\circ$ for which the approximation is applicable.

35. In a thermal equilibrium, the radiative power of a star absorbed by a planet is equal to the power emitted by the planet. [14] The star has a constant radiative power, denoted by P. The irradiance[15] decreases with $\frac{1}{d^2}$, where d is the distance from the star. That's because the total radiant flux through a spherical surface [16] of an arbitrary radius d must be the same, as it cannot disappear.

Only a cross section of the planet of an area πR_c^2 is irradiated, but the planet emits radiation from its entire surface $4\pi R_c^2$. Equating the incoming and outgoing radiative power, we obtain

$$\frac{P}{4\pi D_c^2}\pi R_c^2 = 4\pi R_c^2 \sigma T_c^4 \implies T_c^4 = \frac{P}{16\pi D_c^2} \implies T_c^4 D_c^2 = \mathrm{const.}$$

We observe that the equlibrium temperature of the planet is a function of the distance, but is independent of its size. As the newfound planet orbits the same star as the planet *Cimermanos*, we can use the third Kepler's law,

$$\frac{\tau^2}{D^3} = \mathrm{const.} \implies \frac{\tau_c^2}{D_c^3} = \frac{\tau_n^2}{D_n^3} \implies D_n = D_c\left(\frac{\tau_n}{\tau_c}\right)^{\frac{2}{3}}$$

Using $T^4 D^2 = \mathrm{const.}$, which we obtained earlier, and the third Kepler's law we find the temperature of the newfound planet T_n.

$$T_n = T_c\sqrt{\frac{D_c}{D_n}} = T_c\sqrt{\left(\frac{\tau_c}{\tau_n}\right)^{\frac{2}{3}}}$$

$$T_n = T_c\sqrt[3]{\frac{1}{8}} = \frac{T_c}{/}2$$

[14]The total power radiated by a body is given by Stefan-Boltzmann law.
[15]the radiative power per unit area
[16]Actually, any closed surface containing the star.

$\boxed{36.}$ First of all, let's make it clear why the scales show different value. There are two reasons: first, we are further from the surface of the Earth, therefore the gravitational field is weaker; second, we move with velocity $u = 600\,\mathrm{m\,s^{-1}}$ against the Earth's surface, which is moving towards us at some large speed.

Therefore, we move with larger velocity, which means stronger centrifugal force.[17] The airplane thus moves with velocity v (measuring standing at the earth's surface)

$$v = u + \omega_{Zem} R_Z \doteq 1063.82\,\mathrm{m\,s^{-1}}$$

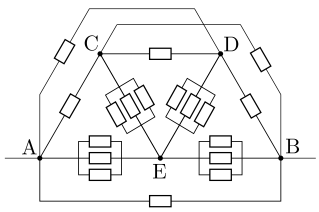
Martin feels both the gravitational and centrifugal force. Therefore, we can expect that he notices a small change in his weight. Precisely, the scales show

$$m^* = \frac{F_G - F_o}{g} = \frac{G\dfrac{M_z m}{(R_z + h)^2} - \dfrac{mv^2}{R_z + h}}{g}$$

From which one could easily obtain Martin's mass:

$$m = \frac{m^* g (R_z + h)}{G\dfrac{M_z}{(R_z + h)} - v^2} \doteq 91.91\,\mathrm{kg}$$

$\boxed{37.}$ The first thing we need to be aware of is that if we have any two points connected with perfect conductor, these points have equal electric potential and no current flows between them. Therefore we can consider them to be a single point. This enables us to rearrange the scheme and to examine if it can be simplified.

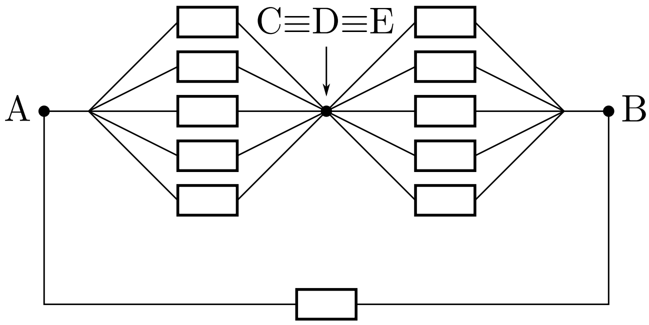


Obr. 34: The first simplification of the tetrahedron

Now we can clearly see that points C, D, and E lie on the axis of symmetry of this resistance scheme. It means they have the same potential and we can again unify them.[18] Now there should not be any problem with reducing of this resistance scheme to the combination of serial and parallel connections of resistors.

[17] We could also look at the motion of the airplane in other frame of reference, but then we would also have to take the effects of Coriolis force into account.

[18] You can find out more about connecting and disconnecting points with the same potential in the fourth chapter of this document `http://fks.sk/~juro/docs/odpory.pdf` (in Slovak).



Obr. 35: The second simplification of the tetrahedron

The result is obtained without cumbersome calculation. We easily obtain resistance of five resistors connected in parallel ($R/5$) and then resistance of the whole tetrahedron

$$R_0 = \frac{\frac{2R^2}{5}}{\frac{2R}{5} + R} = \frac{2}{7}R\,.$$

$\boxed{38.}$ Tom started at altitude h_1. Atmospheric pressure at this altitude was $p_1 = p_0 e^{-(c/p_0)h_1}$. There was air enclosed in the pack of croissant with volume $V_1 - V_0$. It is important to realize that pack can freely change its volume up to maximum volume V_2 and the air inside pack always has the temperature of the air outside. It means that the process inside the pack is isothermal until the pack inflates completely to its maximal volume V_2. The pack reaches its maximum volume at altitude h_2 and that is described by equation

$$p_0 e^{-(c/p_0)h_1}\left(V_1 - V_0\right) = p_0 e^{-(c/p_0)h_2}\left(V_2 - V_0\right)$$

. After some manipulation we get

$$h_2 = \frac{p_0}{c}\left(\ln\frac{V_2 - V_0}{V_1 - V_0}\right) + h_1$$

.

Volume of the pack cannot change after this moment. Moreover, because temperature of the pack is constant, pressure of air inside the pack is constant too. It means that difference between pressure in the pack and pressure outside of the pack occurs during further ascent. Let over-pressure in the altitude h_3 reach the critical value Δp_k. At this moment, the pack bursts. In the moment of bursting, there is still pressure corresponding to the height h_2, so we can write equation for the pressure

$$p_0 e^{-(c/p_0)h_3} + \Delta p_k = p_0 e^{-(c/p_0)h_2}\,.$$

From previous equation we get

$$h_3 = -\frac{p_0}{c}\ln\left(e^{-(c/p_0)h_2} - \frac{\Delta p_k}{p_0}\right)$$

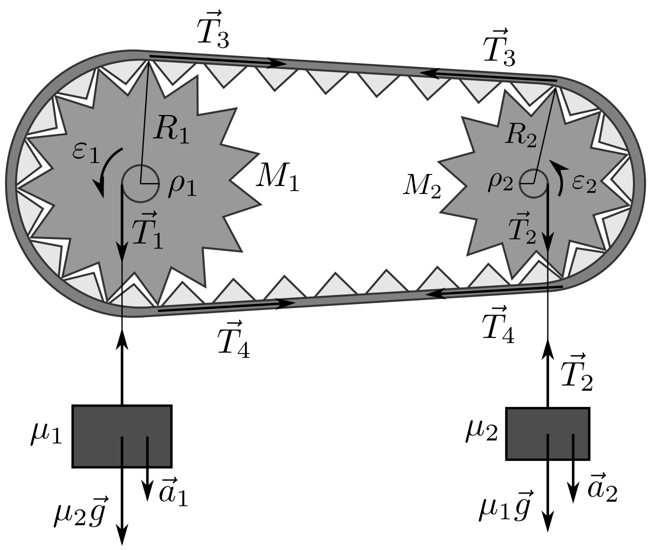
.

After substitution of the expression for h_2 and after subsequent simple manipulation, we obtain the final equation

$$h_3 = -\frac{p_0}{c} \ln \left(\frac{V_1 - V_0}{V_2 - V_0} e^{-(c/p_0)h_1} - \frac{\Delta p_k}{p_0} \right) .$$

Now we can substitute specific values according to the task assignment and we find out that Tom's croissant bursts at altitude 4267 metres above sea level.

39. We're interested in the acceleration a_1 of the first mass μ_1. The system is interconnected via chains and ropes so every part influences the rest. That means we have to analyse the influences of its parts.



Obr. 36

Let's assume that the mass μ_1 moves downwards with acceleration a_1. Therefore the left wheel accelerates clockwise with angular acceleration $\varepsilon_1 = \dfrac{a_1}{\rho_1}$. The chain ensures the coupling of the motion. It is straight all the time and cannot slip through the cogwheels. That implies that the circumferential velocity of both cogwheels is the same. That is true at an any time so circumferential acceleration has to be the same as well. In terms of symbols, $\varepsilon_1 R_1 = \varepsilon_2 R_2$. This consequently determines the angular acceleration of the second cogwheel $\varepsilon_2 = \dfrac{R_1}{R_2}\varepsilon_1$ (clockwise) which also determines the acceleration of mass μ_2. Consequently $a_2 = \varepsilon_2 \rho_2$.

Let's look at the force interaction between the inner parts of our system. There is gravitational force $\mu_1 g$ acting on mass μ_1 as well as tension from the rope T_1. Since the rope is ideal, it does not stretch and has zero mass, the same tension force has to act on the other part of the rope. Consequently, thanks to Newton's third law, the rope acts on the rod and cogwheel with force T_1 downward. The equation of motion for mass is thus

$$\mu_1 a_1 = \mu_1 g - T_1$$

. Analogously,

$$\mu_2 a_2 = T_2 - \mu_2 g$$

.

The only thing remaining is to write down the forces which determines the motion of cogwheels. We already know that reactions to tension force acts on cogwheels with respective torques $T_1\rho_1$, or $T_2\rho_2$. Furthermore, the chain is straight which influences the circumferential velocity of cogwheels, thus there has to be some force from chain acting on the cogwheels. We have to realise that, in general, the force acting on the bottom part of the chain can be different from the force on the upper part of the chain.[19]

But the force acting on the ends of the chain free part (between two contact points) is the same. For the same reason as in the case of the ropes. Let's determine the tension force in chain by T_3, T_4. According to the Third Newton's law the force acting on the chain acts on cogwheel with forces of equal magnitude. The moment of inertia of a disc is $J = \frac{1}{2}MR^2$. Thus the equations of the motion of cogwheels are:

$$\frac{1}{2}M_1 R_1^2 \varepsilon_1 = T_1\rho_1 + T_4 R_1 - T_3 R_1$$

$$\frac{1}{2}M_2 R_2^2 \varepsilon_2 = T_3 R_1 - T_2\rho_2 - T_4 R_1$$

So now we have four equations of motion and three relations between accelerations and angular accelerations. Now we have eight unknown variables, but we are not interested in the forces T_3 a T_4. Only their difference does matter so up to this couple of known variables the solution is uniquely determined. We are interested only in a_1. The solution is left as an exercise. The resulting acceleration is:

$$a_1 = \frac{\mu_1\rho_1 R_2 - \mu_2\rho_2 R_1}{\mu_1 R_2 \rho_1^2 + \frac{1}{2}\left(M_1 + M_2\right) R_1^2 R_2 + \mu_1 \frac{R_1^2\rho_2^2}{R_2}}\rho_1 g \ .$$

We could also consider another way of solving this problem. In the ideal case there are no losses thus the mechanical energy is conserved. This approach, however, requires the ability to differentiate. We have to realise that the only outer force is the gravitational force which is constant (and mass of the system parts does not change also) so we can expect also the constant acceleration. This suggests that we could use only the knowledge of uniform accelerated motion.

Let's think about the motion of the individual parts. One mass moves downwards, the second upwards. Furthermore, the kinetic energy of all parts rises. Let's look at the system in time τ from the moment of releasing the mass. From the conservation of the energy:

$$\Delta E_{p1} = \Delta E_{p2} + E_{k1} + E_{k2} + E_{r1} + E_{r2} \ .$$

During the time τ mass μ_1 decreases by $\frac{1}{2}\varepsilon_1\rho_1\tau^2$ so the change of its potential energy is $\Delta E_{p1} = \mu_1 g \frac{1}{2}\varepsilon_1\rho_1\tau^2$. Analogously the potential energy of mass μ_2 rises by $\Delta E_{p2} = \mu_2 g \frac{1}{2}\varepsilon_2\rho_2\tau^2$.

[19] Since the chain is stiffly connected to the cogwheels, the tension in the chain does not transfer around the circumference of cogwheels, but only on the contact points.

The velocity of mass μ_1 in time τ is $v = \varepsilon_1 \rho_1 \tau$ so its kinetic energy equals $E_{k1} = \frac{1}{2}\mu_1 \left(\varepsilon_1 \rho_1 \tau\right)^2$. Analogously the kinetic energy of mass μ_2 equals $E_{k2} = \frac{1}{2}\mu_2 \left(\varepsilon_2 \rho_2 \tau\right)^2$.

Finally, the rotational energies of cogwheels are $E_{r1} = \frac{1}{2} J_1 \left(\varepsilon_1 \tau\right)^2$ and $E_{r2} = \frac{1}{2} J_2 \left(\varepsilon_2 \tau\right)^2$ where $J = \frac{1}{2} M R^2$ is moment of inertia of cogwheels. The angular accelerations of cogwheels are interconnected via $\varepsilon_1 R_1 = \varepsilon_2 R_2$.

Putting all these together into conservation of energy gives

$$\varepsilon_1 = \frac{\mu_1 \rho_1 R_2 - \mu_2 \rho_2 R_1}{\mu_1 R_2 \rho_1^2 + \frac{1}{2}\left(M_1 + M_2\right) R_1^2 R_2 + \mu_1 \frac{R_1^2 \rho_2^2}{R_2}} g \,,$$

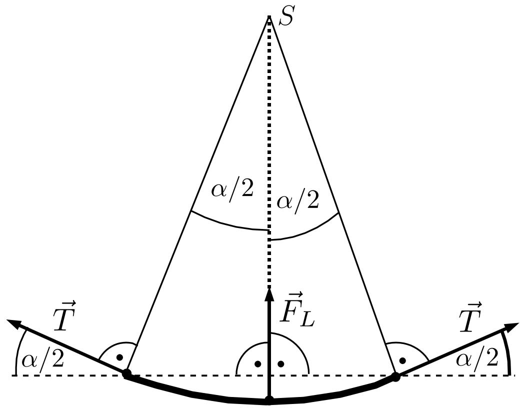
which consequently gives the same a_1 as in the "force" approach.

40. The rope achieves maximal tension in a moment when it snaps. If rope snaps, freely hanging with length L, the maximal tension has the value:

$$T = mg \,,$$

Where m is the mass of rope, which we don't know, but it doesn't matter.

Let's look at the spinning: consider circular loop of radius $R = \frac{L}{2\pi}$ in vertical plane. The problem exhibits circular symmetry, so we can consider any small part of the loop of mass $m^* = m\frac{\alpha}{2\pi}$. The angle α determines the small part of the loop (see figure):



Obr. 37

slucka.eps

When we spin the loop with angular velocity ω the centrifugal force acting on small part of the loop of mass m^* is

$$F_o = \frac{m^* v^2}{R} = m^* \omega^2 R = m\frac{\alpha}{2\pi}\omega^2 \frac{L}{2\pi} = \frac{m\omega^2 L}{4\pi^2}\alpha$$

At the moment of snapping, the small part of the loop is under maximal tension T. The resulting force of tension forces directs in opposite direction to the centrifugal force:

$$F_l = 2T\sin(\alpha/2) \approx T\alpha = mg\alpha$$

Where we use the approximation $\sin\alpha \approx \alpha$ since the angle α is sufficiently small.

By comparing force F_l and F_o we get the result (which is, of course, independent on the rope mass).

$$\frac{m\omega^2 L}{4\pi^2}\alpha = mg\alpha \qquad \longrightarrow \qquad \omega = 2\pi\sqrt{\frac{g}{L}}$$

$\boxed{41.}$ First, we need to know when the speed during collision is maximum. We should definitely use the fact that the Earth orbits the Sun with relatively large velocity. Our dust grains should then orbit the Sun in the opposite direction and have as high velocity as possible.

The orbital velocity of Earth is clearly determined by radius of its orbit and mass of the Sun. We know that force that holds Earth in its orbit is gravitational force:

$$\frac{mv_z^2}{R} = G\frac{mM}{R^2}$$

Hence,

$$v_z = \sqrt{\frac{GM}{R}}.$$

What is then the maximum possible speed of the grains? Their trajectory is parabolic, so they are moving along the escape trajectory with net zero mechanical energy. Hence, the velocity with respect to Sun has to be equal to the escape velocity in the distance in which these grains are currently present.

We need to know their velocity in the moment of the collision, when their distance from Sun is equal to the radius of the Earth's orbit. We could find this distance by use of gravitational potential, but we will rather use the well-known fact that kinetic energy of the body in the circular trajectory is equal to the half of kinetic energy of the body on the escape trajectory in the same place in the gravitational field. [20] If the energy is doubled, the velocity has to be $\sqrt{2}$–times greater, so we designate $v_E = \sqrt{2}v_z$. Therefore, the magnitude of velocity of grains is also determined. All what we need to do now, is to add these velocities, for in optimal case both bodies are moving in the opposite directions.

However, except of this direct collisional velocity, we cannot forget that velocity of the infalling body increases during fall into the Earth's potential well. By how much? Well, overall kinetic energy has to increase by the difference in potential energies of the grain in infinite distance and at the surface of the Earth.

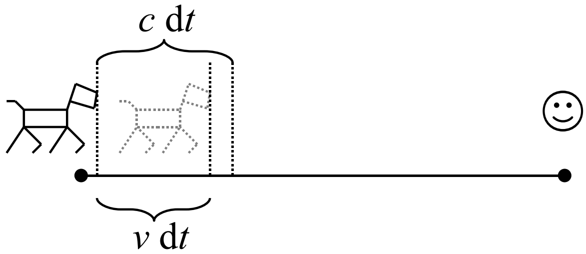
Now, we need to be careful: we do not add velocities, but energies, which are dependent on the squares of velocities.

[20] If you do not believe, try to verify this statement by calculation! :).

Final velocity is therefore square root of sum of squares of both intermediate results.

$$\sqrt{(v_z + v_E)^2 + v_e^2} = \sqrt{\frac{GM}{R} \cdot \left(1 + \sqrt{2}\right)^2 + \frac{2Gm}{r}} = 72{,}879\,\mathrm{m\,s^{-1}}$$

$\boxed{42.}$ Although Jacob and Matthew obviously did not realize many things, they can find out the real velocity even from the apparent velocity v_{real}.



Obr. 38: Photons radiated by Canis Major. The image is adapted from FX Problem Book.

Imagine a photon starts to travel from unknown object (it has nothing to do with UFO) in the direction of our telescope (e. g. from the direction of Canis Major). In a time interval Δt photon travels distance $c\Delta t$. However, an unknown object simultaneously travels the distance $v\Delta t$. If another photon starts its journey to our eye after this time, the spatial distance between these two photons will be $(c - v)\Delta t$. Because we do not know that "photons did not travel all the time at the speed of light"[21], we interpret this information as if the time interval between beginning of the photons' journeys were $\Delta t' = \frac{(c-v)}{c}\Delta t$. It says that our senses and instruments assume that photons "were still moving with speed of light". It means that photons travelled the distance $v\Delta t$ in shorter time interval $\Delta t'$. This phenomenon influences the apparent velocity that we measure,

$$v_{\text{apparent}} = \frac{v_{\text{real}}\Delta t}{\Delta t'} = \frac{c v_{\text{real}}}{c - v_{\text{real}}}$$

For the given values we find the real velocity of unknown object to be $\frac{3}{4}c$.

If the objects moves away from us, photons sent in time interval Δt will be $(c+v)\Delta t$ apart, therefore we will observe apparent velocity of the object to be lower than its real velocity.

$$v_{\text{apparent}} = \frac{c v_{\text{real}}}{c + v_{\text{real}}}$$

In this case, we measure apparent velocity $v_{\text{apparent}} = \frac{3}{7}c$.

[21] It means that one of the photons was just travelling with the unknown object for some time and was waiting to be sent off

Author's note: If this problem caught your attention, you may find its more difficult version in FX Problem Book[22], page 198.

43. By naive comparison of incident and radiated power from black slat, we obtain temperature $T_d = 5334\,\mathrm{K}$.

$$S\sigma T^4 = S/10\sigma T_d^4 \implies T_d = \sqrt[4]{10}T = 5334\,\mathrm{K}$$

. Which is, of course, wrong!

In equilibrium state (in state of thermodynamic equilibrium) are temperatures of bodies that are in equilibrium equal regardless of way of heat transfer (radiation, conduction, convection...). If temperatures in equilibrium state were not equal, we could attach a reversible heat engine to them, which would carry out the work because of transfer of heat from body with higher temperature to body with lower temperature. Temperatures would therefore equalize. Now, we would remove heat engine Because state with equal temperatures is not equilibrium state, temperatures would be stabilized at different values. Now we would re-attach our reversible heat engine... You should now see the problem: by similar process, we could carry out the work only by cooling of our system. Because we are yet to find any*perpetuum mobile*, we believe that it is impossible.

Therefore only solution is $T = T_d = 3000\,\mathrm{K}$.

44. The orbit of the planet is circular, so a centripetal force $F_{cp} = \frac{mv^2}{r}$, where m is planet's mass, must act on the planet. The gravitational force acting on the planet is $F_g = G\frac{mM}{r^2}$. Other than that, we need to account for the flux of particles from the star, which are pushing the planet away.

The power P of the star is radiated into the entire space. The radiant flux density [23] at a distance r from the star is $\frac{P}{4\pi r^2}$. The cross section of the planet is πR^2, so in time τ it absorbs energy $E = \frac{P\pi R^2 \tau}{4\pi r^2}$. The energy of a single photon is $E_0 = hf = \frac{h}{\lambda}$ and its momentum is $p = \frac{h}{\lambda} = \frac{E_0}{c}$. The planet absorbs all photons and therefore its momentum is increased by the momentum of the photons. The total momentum of photons absorbed in time τ is then $\frac{E}{c}$. Using Newton's second law of motion, $F = \frac{\Delta p}{\tau}$, we obtain the force acting on the planet, $F = \frac{PR^2}{4cr^2}$.

As the forces must be in equilibrium, we have

$$\frac{mv^2}{r} = G\frac{mM}{r^2} - \frac{PR^2}{4cr^2}$$

. We find the expression for the orbital velocity using the period T, $v = \frac{2\pi r}{T}$, and obtain

$$\frac{4\pi^2 m}{T^2} r^3 = GmM - \frac{PR^2}{4c}\,.$$

[22] fx.fks.sk

[23] Power per unit area.

Thus

$$r = \sqrt[3]{\frac{T^2}{4\pi^2 m}\left(GmM - \frac{PR^2}{4c}\right)}$$

. Finally, we express planet's mass using its radius, and obtain

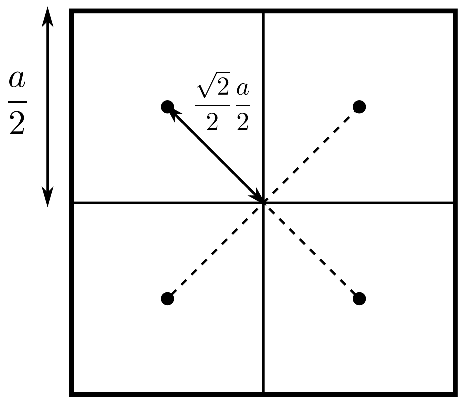
$$r = \sqrt[3]{\frac{T^2}{4\pi^2}\left(GM - \frac{3P}{16\pi c\varrho R}\right)}$$

.

$\boxed{45.}$ The solution of the task is divided into two parts, although the same trick is used in both. The trick is… *scaling*!

Before calculation of moment of inertia of Mike's square, we shall compute the moment of inertia of common square $I(a)$ with side length a and mass M with respect to its center using scaling method. This method is based on the fact that moment of inertia of a square can be expressed as $I = kMa^2$, where k is a numerical constant. Moment of inertia cannot depend on another dimension of square because its side length a is the only parameter that characterizes a square (except of its mass, of course!).

The moment of inertia of the square can be determined if we realize that one large square with side length a can be divided into four smaller squares with side length $a/2$ and we use Steiner theorem! We also know that moment of inertia $I(a/2)$ of square with side length $a/2$ is 16 times smaller smaller than moment of inertia I_a of square with side length a. The reason is that after reducing side length to one half of its initial value, surface area becomes only one fourth of the initial value and so does the mass. Furthermore, distance of any two points of the square is decreased by half, so the distance of all points from the center is halved as well. Because we use square of distance of elements in equation for moment of inertia, we have another decrease to one fourth. After we consider both these effects (decreases in side length and mass) we realize that moment of inertia of smaller square is one sixteenth of that of the original square.



Obr. 39: Use of scaling and Steiner theorem in calculation of moment of inertia of a square

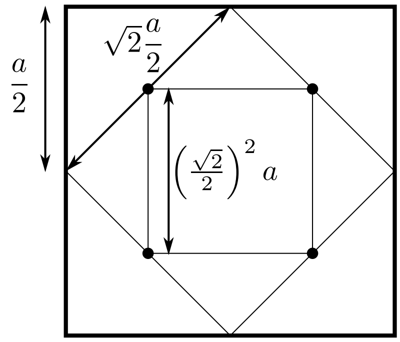
$$I(a) = 4I(a/2) + 4\,(M/4)\left(\frac{\sqrt{2}}{2}\frac{a}{2}\right)^2$$

$$I(a) = 4\frac{I(a)}{16} + 4\,(M/4)\left(\frac{\sqrt{2}}{2}\frac{a}{2}\right)^2$$

$$I(a) = \frac{I(a)}{4} + \frac{1}{8}Ma^2$$

$$I(a) = \frac{1}{6}Ma^2$$

Now we calculate the moment of inertia of Mike's square, which is fractal. Fractals display a beautiful property: They look the same after any magnification (roughly speaking). We use this property again in the same trick. Total moment of inertia of Mike's square I_{MS} can be calculated as a moment of inertia of square with side length a. We then subtract the moment of inertia of square with smaller side length (cut) and, subsequently, we add moment of inertia of Mike's square with smaller side length. All we need to do now is to find lengths of sides of particular squares.



Obr. 40: Mike's fractal square

$$I_{\mathrm{MS}}(a) = I_{\mathrm{square}}(a) - I_{\mathrm{square}}\left(\frac{\sqrt{2}}{2}a\right) + I_{\mathrm{MS}}\left(\left(\frac{\sqrt{2}}{2}a\right)^2\right)$$

$$I_{\mathrm{MS}}(a) = I_{\mathrm{square}}(a) - \left(\frac{\sqrt{2}}{2}\right)^4 I_{\mathrm{square}}(a) + \frac{1}{16}I_{\mathrm{MS}}(a)$$

$$I_{\mathrm{MS}}(a) = \frac{4}{5}I_{\mathrm{square}}(a) = \frac{2}{15}Ma^2$$

$\boxed{46.}$ Let's start by analysing how Patrick's wheel had been made. The original wheel had mass M_0 and there was a hole drilled into it. Therefore the mass of the wheel decreased by m.

The surface mass density is obviously constant across the whole wheel. Thus the mass M_0 can be found just by simple comparison of areas: $M_0 = M\dfrac{R^2}{R^2 - r^2} = \dfrac{9}{8}M$ a $m = M\dfrac{r^2}{R^2 - r^2} = \dfrac{1}{8}M$.

With this information we can calculate the position of the centre of mass. We can expect that it lies on the axis of symmetry, but we are especially interested in its distance from the centre.

$$y = \frac{\frac{9}{8}M \cdot 0 + \frac{-1}{8}M \cdot \frac{2}{3}R}{\frac{9}{8}M - \frac{1}{8}M} = -\frac{R}{12}$$

We should be concerned about the sign. It just says that the centre of mass is on the opposite side than the hole. Let's determine the distance between the centre of mass and the centre of wheel by $t = -y = \dfrac{R}{12}$.

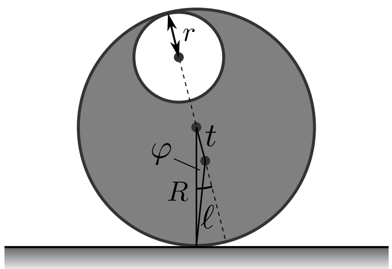
Now, let's look at the motion of Patrick's wheel. If we rotate the wheel by an angle φ, its potential energy increases. Thus

$$E_p = Mgt(1 - \cos\varphi)\,.$$

Kinetic energy can be express as kinetic energy of energy of rotation around instantaneous axis of rotation. But we must not forget that the axis of rotation changes over time.

$$E_k = \frac{1}{2}I_A\omega^2,$$

where I_A stands for the moment of inertia around instantaneous axis of rotation. The distance between the centre of mass and this axis is l which can be calculated using cosine rule $l^2 = R^2 + t^2 - 2Rt\cos\varphi$.



Obr. 41: Geometry of Patrick's wheel

The only unknown variable is the moment of inertia of Patrick's wheel with respect to the rotational axis. Let's start by calculating the moment of inertia around the axis passing through the centre of mass. Using parallel axis theorem and the additivity of moment of inertia we can then calculate the moment of inertia of Patrick's wheel. [24] Thus

$$\frac{1}{2}\frac{9}{8}MR^2 = \left(I_T + Mt^2\right) + \left(\frac{1}{2}\frac{1}{8}Mr^2 + \frac{1}{8}M\left(\frac{2}{3}R\right)^2\right)\,.$$

[24] The moment of inertia of a disc is $\frac{1}{2}mr^2$.

From this equation we get $I_T = \frac{71}{144}MR^2$. Using parallel axis theorem one more time we get the moment of inertia around the instantaneous axis of rotation:

$$I_A = I_T + Ml^2 = \frac{71}{144}MR^2 + M\left(R^2 + t^2 - 2Rt\cos\varphi\right) .$$

We are obviously interested only in small oscillations therefore $\cos\varphi \approx 1$ [25] and moment of inertia is thus $I_A = \frac{4}{3}MR^2$.

The energy of Patrick's wheel can be calculated by approximation

$$(1 - \cos\varphi \approx \frac{\varphi^2}{2}) .$$

Therefore

$$E = \frac{1}{2}Mg\frac{R}{12}\varphi^2 + \frac{1}{2}\frac{4}{3}MR^2\omega^2 = const.$$

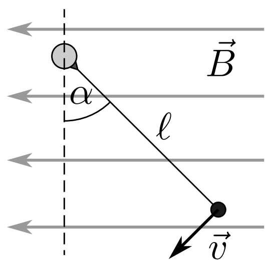
Where we clearly see a harmonic oscillator with period determined by constant next to φ^2 and ω^2. Thus the period of small oscillations is

$$T = 2\pi\sqrt{\frac{\frac{4}{3}MR^2}{Mg\frac{R}{12}}} = 8\pi\sqrt{Rg} .$$

$\boxed{47.}$ Once released, the rod begins to oscillate like a swing. According to Faraday's law of electromagnetic induction, voltage is induced between the ends of a conductor, which depends on the length of the conductor l, its velocity $\mathbf{v}$ and the magnetic flux density $\mathbf{B}$. Its magnitude is:

$$U = lvB\sin\alpha ,$$

where α is the deflection of the frame from vertical. [26] When does the voltage reach maximum? We know for sure that it's neither at the beginning (because the velocity is 0), nor in the lowest position (the sine is 0). It is therefore somewhere between these values.



Obr. 42

[25] This is the consequence of Taylor series of $\cos\varphi$ at $\varphi = 0$

[26] Or more precisely, it is the angle between the velocity vector $\mathbf{v}$ and the vector of magnetic flux density $\mathbf{B}$.

Velocity v and angle α are linked by the conservation of energy:

$$mgl\cos\alpha = \frac{1}{2}mv^2\,, \qquad \to \qquad v = \sqrt{2lg\cos\alpha}$$

Plugging this into the equation for voltage, we get $U = Bl\sqrt{2lg}\sqrt{\cos\alpha}\sin\alpha$. We want to know at what angle α does the voltage reach its maximum, thus we take a derivative:

$$\begin{aligned}\frac{\partial U}{\partial\alpha} = Bl\sqrt{2lg}\left(\cos\alpha\sqrt{\cos\alpha} - \frac{\sin^2\alpha}{2\sqrt{\cos\alpha}}\right) &\stackrel{!}{=} 0\\ 2\cos^2\alpha &= \sin^2\alpha\\ \cos\alpha &= \frac{1}{\sqrt{3}} \qquad \to \qquad \sin\alpha = \frac{\sqrt{2}}{\sqrt{3}}\,,\end{aligned}$$

which corresponds to angle $\alpha = 54.74°$. We can now get the largest voltage U_{max} by plugging in this angle:

$$U_{max} = Bl\sqrt{2lg}\sqrt{\cos\alpha}\sin\alpha = Bl\sqrt{2lg}\frac{1}{\sqrt[4]{3}}\frac{\sqrt{2}}{\sqrt{3}} = \sqrt[4]{\frac{16}{27}}\sqrt{gB^2l^3}$$